Combinatorics of Syzygies

Toric Resolutions
Wednesday, May 25, 4:00pm
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Exercise 1. Let $A \in \mathbb{N}^{d\times n}$ were we assume that the first row of A is $(1, 1, \ldots, 1)$. Using the `Toric` command compute a Gröbner basis and a minimal generating set for the toric ideal I_A for various matrices A. Try 2×4, 2×5, 3×5 and 3×6 matrices.
Next develop a `CoCoA` function called `ToricFiber` that takes as an input a toric ideal I_A and a monomial x^u and computes the fiber that x^u belongs to, i.e. $\{x^v : Av = Au\}$. (Hint: let $\mathcal{G}$ be a Gröbner basis I_A. Given x^u you can compute the unique normal for of x^u with respect to $\mathcal{G}$.)

Exercise 2. Generate random $d \times (d+2)$ matrices and their corresponding codimension two toric ideals I_A. For each instance compute the minimal free resolution of S/I_A, and compute the fibers of the multidegrees the syzygies belong to. Catalog your results. Any patterns?

Exercise 3. Generic toric ideals are, in principal, abundant, but it is still not so easy to generate them. Let's look at the case where $A = [a_1, a_2, a_3, a_4]$ where $a_1 < a_2 < a_3 < a_4$. The first example appears when $a_1+a_2+a_3+a_4 = 100$. Make an exhaustive search for the cases when $101 \leq a_1 + a_2 + a_3 + a_4 \leq 150$. To make the computations more efficient discard the cases when $gcd(a_1, \ldots, a_4) \neq 1$ and when one a_i divides another a_j. For each instance you find compute and draw the Scarf complex of a reverse lexicographic initial ideal.

Exercise 4. Repeat Exercise 3 with randomly chosen a_1, a_2, a_3, a_4. How much can you push your computations? How many minimal generators do you get?

Exercise 5. Prove the following two statements:

a) A generic monomial ideal I is Cohen-Macaulay if and only if Δ_I is pure.
b) A generic monomial ideal I has the *chain property*: if P is an associated prime of I then there is a chain of associated primes $P = P_0,\ P_1,\ P_2, \ldots, P_k = Q$ where Q is a minimal prime of I and $\mathrm{d}im(P_{i+1}) = \mathrm{d}im(P_i) + 1$ for all $i = 0, \ldots, k-1$.

Exercise 6. Compute the Scarf complexex of all the generic deformations of I^3 where $I = \langle x,\ y,\ z\rangle$. Also develop a *CoCoA* function *MakeGeneric* that will deform a given monomial ideal to a generic monomial ideal. Can you modify your code in Exercise 4 to compute the irredundant irreducible decomposition of a (not necessarily generic) monomial ideal I?

Exercise 7. In Lecture 4 we saw that a generic monomial ideal I in $k[x, y, z, w]$ where every pair of generators form an edge of Δ_I can have at most 12 minimal generators. Can you give two such extremal monomial ideals with non-isomorphic Scarf complexes? A more difficult question: can you classify all Scarf complexes of these extremal monomial ideals?