Combinatorics of Syzygies

Monomial resolutions and squarefree ideals
Monday, May 23, 4:00pm

Anna Bigatti and Serkan Hoşten

Exercise 1. Develop a `CoCoA` function called `GetMultidegrees` that takes as an input a monomial ideal I and computes the $\mathbb{N}^n$-graded degrees of the syzygies in a minimal free resolution of S/I. The output should be a list of lists, one for each matrix in the free resolution.
Now use `CoCoA` to generate random monomial ideals in 4 variables x_1, x_2, x_3, x_4 (with five to eight minimal generators) such that for any two minimal generators x^a and x^b one has $a_i \neq b_i$ for all i unless $a_i = b_i = 0$. Compute the minimal free resolutions and use `GetMultidegrees` to list the $\mathbb{N}^n$-degree of the syzygies. Make a conjecture about the minimal first syzygies and their degrees. Can you generalize your conjecture to the degrees of the higher syzygies?

Exercise 2. Write a `CoCoA` function `StanleyReisner` that takes a simplical complex Δ as a list of its facets and computes I_Δ. Conversely, write a function `IdealToComplex` that will compute the associated simplicial complex for a given squarefree monomial ideal.

Exercise 3. Develop a `CoCoA` function `AlexanderDual` that computes the Stanley Reisner ideal of Δ^* of a simplicial complex Δ.

Exercise 4. Describe how one can use `CoCoA` to compute the homology of a simplicial complex Δ. Test your idea (using `AlexanderDual` and `GetMultidegrees`) on the boundary of a triangle, a tetrahedron, and an octahedron. Generate random 2-dimensional simplical complexes on 8 vertices, draw them, and test again your idea.

Exercise 5. This exercise illustrates that there are still many questions one can ask and hopefully answer about squarefree monomial ideals. The observations that we want you to make have been made very recently.
Let G be a graph on n vertices. A *clique* of G is a subset W of the vertices such that every vertex in W is connected to any other vertex in W by an edge of G. The *clique complex* of G is the simplicial complex on n vertices where the faces are the cliques of G. First prove that a squarefree monomial ideal I is generated by degree 2 generators if and only if I is the Stanley-Reisner ideal of the clique complex of a graph.
Now some more definitions: a *cycle* of length k in a graph G is a collection of the edges $\{v_0, v_1\}$, $\{v_1, v_2\}, \ldots, \{v_{k-2}, v_{k-1}\}, \{v_{k-1}, v_0\}$. A *chord* of a k-cycle is an edge $\{v_i, v_j\}$ where $j \neq i+1$ and $j \neq i-1 \ \mathrm{m}odk$.
First, construct graphs on 10 vertices with no cycles (these graphs are called *trees*) as well as graphs where the only *chordless* cycles are triangles. Compute the free resolution of the Stanley-Reisner ideal of the clique complex of these graphs. What is interesting about the matrices in these free resolutions?
Now repeat the same with graphs where the shortest chordless cycle that is not a triangle has length k with $k = 4, 5, 6$. What do you see now? Make a conjecture!

Exercise 6. In this exercise we will look at the h-polynomial of S/I_Δ where Δ is the boundary complex of a simplical polytope. First generate some three-dimensional simplicial polytopes and compute their h-polynomial using the `Poincare` command. What do you see? Confirm your observation on some four and five dimensional simplicial polytopes.
In general, let Δ be a $(d-1)$-dimensional shellable simplical complex with shelling $F_1, \ldots, F_m$. For $j = 2, \ldots, m$, let r_j be the number of facets of the intersection of F_j with $\bigcup_{i=1}^{j-1} F_i$, and set $r_1 = 0$. Define $h_i = |\{j : r_j = i\}|$ for $i = 0, 1, \ldots, d$. Using a line shelling for the simplical polytopes you have constructed compute the numbers h_i in each case and compare to the h-polynomials. What is the obvious conjecture? How much can you prove it?