A Computational Introduction to the Weyl Algebra and D-modules

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1 Introduction

The algebraic D-modules theory is related with the study of modules over the Weyl Algebra. *Why D-modules?*, as S. C. Coutinho points in his splendid book [16], is a particularly easy to answer question. Hardly any area of Mathematics has been left untouched by this theory: from Number Theory to Mathematical Physics and from Singularity Theory to Representation of Algebraic Groups, to mention only a bunch. Indeed, the theory of D-modules sits across the traditional division into Algebra, Analysis and Geometry and this fact gives to the theory a rare beauty.

1.1 (Very) Brief historical tour

The interest of the Weyl algebra started when a number of people like Heisenberg, Dirac or Born (1925) were trying to understand the behaviour of the atom, and dynamical variables that did not commute were introduced. Weyl's pioneer book *The theory of groups and quantum mechanics* was perhaps its amazing debut in society. Then Littlewood (1933) used the language of infinite dimensional algebras to describe the objects, and Dixmier (1963) connected the Weyl algebra with the Theory of Lie Algebras.

Of course, a natural environment for the Weyl algebra is the study of systems differential equations –in this context the theory is often called *Algebraic Analysis*– considering an equation as a module over a ring of differential equations. This approach comes from people like Malgrange and Kashiwara (see for example [32] and [24], [25]) and, at the same time, from Bernstein[1] (see [6]).

The theory of D-modules can be studied under the analytic or algebraic point of view, depending on the base variety. Highly sophisticated machinery (to begin with, derived categories and sheaves) is needed for the analytic counterpart of the theory and this approach will be out of the scope of these humble notes. Nevertheless, any introduction to this subject has to mention (at least) the spectacular *Riemann-Hilbert correpondence*, obtained at the same time by Kashiwara and Mebkhout (see [26] and [33],[34]).

[1] He developed this theory to give an elementary new answer of a classic problem proposed by Gelfand in the International Congress of Mathematics in 1954 about the extension of a certain complex function. The old proofs used Hironaka's resolution of singularities.

The effective methods started with the works of J. Briançon, Ph. Maisonobe ([9]) and F.J. Castro-Jiménez ([11]) who adapted the theory of the Gröbner bases to this context. As in many other branches of Mathematics, this computational approach has taken a major role as the machines have been able to run efficiently their algorithms.

In recent years, the works of T. Oaku (see [43] to begin with) and his collaborators have given to this branch a substantial push. The most remarkable by far is the celebrated work of T. Oaku and N. Takayama, [44], in which algorithms to compute the main operations for the Weyl algebra were presented. A good list of references can be obtained in [53] and we have tried to include the more actual ones in the bibliography.

Although the algorithms for D-modules need to be improved in the future to treat difficult examples, they have definitively given crucial tools to understand and solve classical and still-open problems.

1.2 References

As well as the cited book of Coutinho ([16]), the books of Björk ([7],[8]) are usual theoretical introductions to the subject. Their lists of references are very complete.

From the computational point of view [53] is an excellent introduction. The theory of Gröbner bases is applied to the study of systems of multidimensional hypergeometric partial differential operators, the so called *GKZ systems* —to pay honour to Gel'fand, Kapranov and Zelevinsky who introduced the subject in the 1980's —. Using the algebraic analogue to classical perturbation techniques in analysis, many problems are reduced to commutative monomial ideals. At the same time, the mentioned book introduces the main new algorithms (the majority of them for holonomic modules) for dealing with rings of differential operators discovered and implemented in recent years.

Finally, as we have mentioned in the introduction, [44] plays a very important role and it can be considered as an excellent starting point to study algorithms for D-modules.

1.3 Packages

In our opinion, the most important available packages for working with D-modules are:

- The *D-module package for Macaulay 2* (see [18]) written by A. Leykin and H. Tsai. It is powerful, user friendly and contains many predefined functions to calculate the interesting issues (b-functions, dimensions, cohomological objects, free resolutions,...).

- The very promising new *Plural/Singular* written by V. Levandovskyy (see [19] and [28]), with the very well known capabilities of Singular and the possibility of computing in the more general context of Poincaré-Birkhoff-Witt algebras.

Some intractable problems in the Weyl algebra have been solved in a slightly different context (see 4.2) using this system.

- The amazing *Risa/Asir system* (see [40]) written by Noro et al., that is able to manage intractable problems for Macaulay 2 too. It can be taken as a whole with the system *kan/sm1* (see [55]) designed by N. Takayama.

- And last , but not least, the new *CoCoA 5* (see [10]) written by the CoCoA team in Genova, which has joined this noble family. We hope that, with the wonderful heritage of the CoCoA's new design, it has important things to say in the future.

2 The Weyl algebra and its basic properties

In this section we will define the Weyl Algebra and present its basic properties: it is a domain, simple and noetherian. Finally, we will consider the modules over the Weyl algebra to define the dimension.

2.1 The Weyl Algebra

Let k be a field of characteristic 0.

Definition 2.1.1 *The* n-th Weyl algebra $A_n(k)$ *is the non commutative free associative algebra* $k\langle x_1,\dots,x_n,\partial_1,\dots,\partial_n\rangle$ *over the (two-sided) ideal generated by the elements*

$$\begin{array}{l} \partial_i x_i - x_i\partial_i - 1,\ i=1,\dots,n,\\ \partial_i x_j - x_j\partial_i,\ 1\le i\ne j\le n\\ x_j x_i - x_i x_j,\ 1\le i,j\le n\\ \partial_j\partial_i - \partial_j\partial_i,\ 1\le i,j\le n. \end{array}$$

We will denote $x=(x_1,\dots,x_n),\partial=(x_1,\dots,\partial_n)$ and $A=A_n(k)$ if there is no confusion.

Lemma 2.1.2 *We have*

$$\partial_i^\alpha x_i^\beta = \sum_{k=0}^{\min\{\alpha,\beta\}} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)\beta(\beta-1)\cdots(\beta-k+1)}{k!} x^{\beta-k}\partial^{\alpha-k}.$$

Proposition 2.1.3 *The set* $\mathcal{B}=\{x^\alpha\partial^\beta,\ \alpha,\beta\in\mathbf{N}^n\}$ *is a basis of* A *as a* k*-vector space.*

An element $P\in A$ is said to be written in *normal form* if it is expressed with respect to the basis $\mathcal{B}$. $\mathcal{B}'=\{\partial^\beta x^\alpha,\ \alpha,\beta\in\mathbf{N}^n\}$ is a basis too.

There is an alternative definition of the Weyl algebra as a subalgebra of the k-linear endomorphisms over $k[x_1,\dots,x_n]$. Both definitions do not coincide if k has positive characteristic.

2.2 First properties

Definition 2.2.1 *The degree of $P \in A$, $P = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha \partial^\beta$ is*

$$deg(P) = \max\{|\alpha| + |\beta|, \text{ where } c_{\alpha,\beta} \neq 0\}.$$

Lemma 2.2.2 *Let P, Q be elements of A. We have:*

- $deg(P + Q) \leq \max\{deg(P), deg(Q)\}$.
- $deg(PQ) = deg(P) + deg(Q)$.
- $deg([P, Q]) \leq deg(P) + deg(Q) - 2$

As two easy consequences we have

Proposition 2.2.3 *A is a domain.*

Proposition 2.2.4 *A is simple. In particular, every endomorphism of A is injective and there are no non-trivial two-sided ideals.*

From now on we will work only with *left* ideals in A (see why is enough in section 2.4).

2.3 The Weyl algebra is noetherian

It is very useful to manage the concept of *homogeneous operator*. Due to the non commutativity of A, we will need the concept of *filtrations* over a k-algebra in order to do so. Associated graded algebras are obtained in this way.

Definition 2.3.1 *The* Bernstein filtration *of A is the increasing sequence F of vector subspaces F_j of A:*

$$F_j = \left\{ \sum c_{\alpha,\beta} x^\alpha \partial^\beta \text{ such that } |\alpha| + |\beta| \leq j \right\} \text{ for } j \in \mathbf{Z}.$$

Clearly, the Bernstein filtration verifies the needed properties:

- $\bigcup_{j \geq 0} F_j = A$.
- For every $i, j \geq 0$ we have $F_i F_j \subset F_{i+j}$.

For the Bernstein filtration the last inclusion is an equality. In addition, F verifies that $F_k = \{0\}$ if $k < 0$, $F_0 = k$. Furthermore the F_j have finite dimension. Once we have the filtration we can defined the correspondent graded algebra $gr_F(A)$,

$$gr_F(A) = \bigoplus_{i \geq 0} F(i) = \bigoplus_{i \geq 0} \frac{F_i}{F_{i-1}}.$$

Definition 2.3.2 *Given $P \in A$, $P \neq 0$, $P \in F_s \backslash F_{s-1}$, s is the* order *of P with respect to F. The* symbol *of P with respect to F, $\sigma_F(P)$, is*

$$\sigma_F(P) = P + F_{s-1} \in gr_F(A).$$

The order of $P = 0$ is $-\infty$ and $\sigma(0) = 0$.

The terminology *initial part of P with respect to F*, $in_F(P)$ instead of the symbol of P is usual too, and we will adopt it in the next section. With the above definitions, there is a canonical isomorphism from A to $gr_F(A)$ as vector spaces: associate to $P \in A$ the sum of its homogeneous components in $gr_F(A)$. We have

Proposition 2.3.3 *$gr_F(A)$ is canonically isomorphic to a ring of polynomials $k[x_1, \ldots, x_n, \xi_1, \ldots,$*

Now it is easy to deduce the main result of this section:

Proposition 2.3.4 *The ring A is a left (resp. right) noetherian ring, i.e. every left (resp. right) ideal is finitely generated.*

2.4 Modules over the Weyl algebra

There is an antihomomorphism ϕ between the category of left and right A-modules defined as follows:

1. $\phi(\lambda) = \lambda$ if λ in k
2. $\phi(x_i) = x_i$ for $i = 1, 2, \ldots, n$
3. $\phi(\partial_j) = -\partial j$ for $j = 1, 2, \ldots, n$,

and by recurrence $\phi(PQ) = \phi(Q)\phi(P)$ for any $P, Q \in A$. Thus it is only necessary to study left A-modules[2].

Given an A-module $\mathcal{M}$ and given a filtration F for A, you can consider a filtration Γ and the correspondent graded $gr_F(A)$-module $gr_\Gamma(\mathcal{M})$. As in A you can naturally define the order and the symbol of an element $m \in \mathcal{M}$.

In order to define the concept of *dimension* of an A-module we take the Bernstein filtration F for A and what is called a *good filtration* Γ with respect to F for $\mathcal{M}$: Γ is a good filtration for $\mathcal{M}$ if $gr_\Gamma(\mathcal{M})$ is a finitely generated $gr_F(A)$-module. The existence of such a filtration is equivalent to the condition for $\mathcal{M}$ to be finitely generated.

Definition 2.4.1 *Let $\mathcal{M}$ be a finitely generated A-module and Γ a good filtration with respect to F, the* Hilbert-Samuel *polynomial associated to $(\mathcal{M}, \Gamma)$, $P(t, \mathcal{M}, \Gamma)$ is the correspondent Hilbert-Samuel polynomial $P(t)$ of the finitely generated graded module $gr_\Gamma(\mathcal{M})$ over the ring of polynomials $gr_F(A) = k[x, \xi]$. The degree of $P(t)$ —that does not depend on the chosen Γ— is the* dimension *of $\mathcal{M}$, $d(\mathcal{M})$.*

One of the biggest differences between the modules over the polynomials and the modules over A is the theorem of Bernstein:

Theorem 2.4.2 *If $\mathcal{M} \neq 0$ is a finitely generated A-module, then $d(\mathcal{M}) \geq n$.*

The modules whose dimension is equal to n are called *holonomic A-modules*.

[2] It is the classical option: the endomorphisms are usually written acting on the left.

3 Gröbner bases in the Weyl algebra

A real vector $(u, v) = (u_1, \ldots, u_n, v_1, \ldots, v_n)$ is a weight vector for the Weyl algebra if $u_i + v_i \geq 0$ for $i = 1, 2, \ldots, n$. Generalizing the results of the previous section, it can be defined the associated graded algebra $gr_{(u,v)}(A)$ with respect to the filtration defined by

$$F_m^{(u,v)} = \{ \sum_{u\alpha + v\beta \leq m} c_{\alpha\beta} x^\alpha \partial^\beta \}.$$

Note that $gr_{(u,v)}(A)$ is not always a commutative algebra. For $P \in A$, $P \neq 0$ we define $in_{(u,v)}$ in the natural way.

Definition 3.0.3 *Let I be an ideal in A and (u, v) any weight vector. The ideal $in_{(u,v)}(I) := \langle in_{(u,v)}(P) | P \in I \rangle \subset gr_{(u,v)}(A)$ is the* initial ideal *of I with respect to (u, v). A finite subset G of I is a* Gröbner basis *of I with respect to (u, v) if I is generated by G and $in_{(u,v)}(I)$ is generated by $\{in_{(u,v)}(P) | P \in I\}$.*

To compute a Gröbner basis we need to define a *multiplicative monomial order* $\prec$, that is,

1. $1 \prec x_i \partial_i$ for $i = 1, 2, \ldots, n$.
2. $x^\alpha \partial^\beta \prec x^a \partial^b$ implies $x^{\alpha+s} \partial^{\beta+t} \prec x^{a+s} \partial^{b+t}$ for every $(s, t) \in \mathbf{N}^{2n}$.

A multiplicative monomial order $\prec$ is a *term order* if 1 is the least element with respect to $\prec$. A *non term order* has infinite strictly decreasing chains. For the most frequently used term orders in the commutative setting see [1] or [27].

Once you have fixed a multiplicative monomial order $\prec$, the *initial monomial $in_\prec(P)$ of an element* $P \in A$ is the largest monomial with respect to $\prec$ in the normal form of P. In the same way for any ideal $I \subset A$

$$in_\prec(I) = \{in_\prec(P) | P \in I\}.$$

Here, the concept of Gröbner basis with respect to $\prec$ is absolutely analogous to the case of weight vectors. The relationship between both concepts is straightforward: if $(u, v) \in \mathbf{R}^{2n}$ is a weight vector and $\prec$ is a term order, then we naturally define a new multiplicative monomial order $\prec_{(u,v)}$ as follows:

$$x^\alpha \partial^\beta \prec_{(u,v)} x^a \partial^b \Longleftrightarrow \alpha u + \beta v < au + bv \text{ or } \alpha u + \beta v = au + bv \text{ and } x^\alpha \partial^\beta \prec x^a \partial^b.$$

The new order is a term order if and only if (u, v) is a non-negative vector. The important theorem is

Theorem 3.0.4 *Let $I \subset A$ be an ideal, (u, v) a weight vector and $\prec$ a term order. If G is a Gröbner basis for I with respect to $\prec_{(u,v)}$ then*

1. *G is a Gröbner basis for I with respect to (u, v).*
2. *$in_{(u,v)}(G)$ is a Gröbner basis for $in_{(u,v)}(I)$ with respect to $\prec$.*

So the problem of computing Gröbner bases with respect to weight vectors has been reduced to the calculation of Gröbner bases with respect to multiplicative monomial orders. The non term orders need a new construction, the *homogenised Weyl algebra* where $\partial_i x_i = x_i \partial_i + h^2$, for h a new variable that commutes with the rest, in order to assure the finiteness of the computations. This idea of using a Rees algebra appeared first in [12]. It has very important applications in many algorithms for the Weyl algebra[3].

If we have a term order the situation is very similar to the commutative case: we have a *division algorithm* that produces a standard representation of any $P \in A$ in terms of a Gröbner basis G, *S-pairs* of two elements of A with multipliers chosen to cancel the initial monomials . The *Buchberger algorithm* is correct with the same S-pair criterium to finish. You can consider *reduced Gröbner basis* too.

Remark 3.0.5 *The reader shouldn't think that* absolutely all *the technical details of the Buchberger algorithm and Gröbner bases for the commutative case are applicable for A. As a sample note that the coprimality test (to accelerate the Buchberger algorithm) (see [1] or [27] for example) is no longer valid in A.*

Exercise: In $A = A_2(\mathbf{C})$, compute the Gröbner basis with respect to the weight vector $((1,1),(1,1))$ of the ideal I generated by the elements

$$3x_1\partial_1 + 2x_2\partial_2 + 6, \quad 3x_2\partial_1 + 2x_1\partial_2.$$

4 Applications

We will treat in these notes three computational techniques to study three problems in D-module theory:

1. Computing the characteristic variety and the dimension of the module A/I, where I is an ideal of A.
2. The computation of the formal annihilator of f^s, where $f \in \mathbf{C}[x_1, \ldots, x_n]$ and the Bernstein-Sato polynomial of f.
3. Logarithmic approximations of $Ann_A(1/f^\alpha)$ where $-\alpha$ is the least integer root of the Bernstein-Sato polynomial of f.

The first two problems are classical. The third is close to the field of research of the authors.

[3] Add [2] or [3] to your list of readings in computational D-module theory.

4.1 Application 1: Testing holonomicity.

Let $I \subset A$ be an ideal. The *characteristic ideal* of $\mathcal{M} = A/I$ is the ideal

$$cch(\mathcal{M}) = Ann(gr_\Gamma \mathcal{M})$$

where Γ is a good filtration (see chapter 11 of [16] to see that the definition does not depend on the chosen good filtration). The *characteristic variety* of $\mathcal{M}$ is the affine variety

$$Ch(\mathcal{M}) = \mathcal{V}(cch(\mathcal{M})) \subset \mathbf{C}^{2n},$$

that is the zeros locus of the characteristic ideal. It is an important invariant of $\mathcal{M}$.

It is a result of Oaku that

$$cch(\mathcal{M}) = in_{(0,e)}(I),$$

where $(0,e) = ((0,\ldots,0),(1,\ldots,1))$. By theorem 3.0.4, it is enough to calculate a Gröbner basis of I with respect to the weight vector $(0,e)$.

The dimension of $\mathcal{M}$ coincide with the dimension of its characteristic variety.

Exercise.- Let $A = A_4(\mathbf{C})$. Calculate the characteristic variety of $\mathcal{M} = A/I$ and test if $\mathcal{M}$ is holonomic for the ideal I generated by the following four operators:

$$\partial_2\partial_3 - \partial_1\partial_4,\ x_1\partial_1 - x_4\partial_4 + 1 - 1,\ x_2\partial_2 + x_4\partial_4 + 1,\ x_3\partial_3 + x_4\partial_4 + 2.$$

4.2 Application 2: Calculation of $Ann_{A[s]}(f^s)$ and the Bernstein-Sato polynomial

Let f be a polynomial in $\mathbf{C}[x] = \mathbf{C}[x_1,\ldots,x_n]$ and $A = A_n(\mathbf{C}) = \mathbf{C}[x,\partial_x]$. Let us consider the algebra $A[s] = A \otimes_{\mathbf{C}} \mathbf{C}[s]$ with the trivial action of the elements of $\mathbf{C}[s]$.

The *Bernstein-Sato polynomial* or *global b-function* of f, $b_f(s)$, is the generator of the principal ideal of the elements $b(s) \in \mathbf{C}[s]$ such that

$$b(s) \cdot f^s = P \bullet f^{s+1}, \quad \text{for some } P \in A.$$

One possible way of computing $b_f(s)$ —it can be derived from [43]— is the following algorithm, divided in two steps:

Step A.- Calculation of $Ann_{A[s]}(f^s)$: Consider the new ring $A[u,v,t,\partial_t]$, where t,∂_t is a new Weyl Algebra pair of variables and u,v conmute. Then:

1.- Calculate the intersection $I_f \cap A[t,\partial_t]$ (using any elimination order with u,v greater than the rest), where

$$I_f = \langle 1 - uv,\ tu - f,\ \partial_i + \frac{\partial f}{\partial x_i} v\partial_t \text{ for } i = 1,2,\ldots,n\rangle.$$

2.- Each of the generators of the ideal computed in 1.- is of the form

$$t^a \cdot p(x,\partial_x,t\partial_t) \cdot \partial^b.$$

Replace each one by

$$[t\partial_t]^a \cdot p(x, \partial_x, t\partial_t - b) \cdot [t\partial_t]_b \in A[t\partial_t].$$

3.- Replace $t\partial_t$ by $-s-1$ in each of the operators computed in 2. The output obtained is $Ann_{A[s]}(f^s)$.

Step B.- Compute $(Ann_{A[s]}(f^s) + \langle f \rangle) \cap \mathbf{C}[s]$ (using again an elimination order $\prec$ with $x, \partial_x \succ s$). The output is a principal ideal whose generator is $b_f(s)$.

Exercise.- Compute the annihilator of f^s and the global b-function of f for the following cases:

- $f = x^2 + y^3 \in \mathbf{C}[x, y]$.
- $f = x^2 - y^3 \in \mathbf{C}[x, y]$.
- $f = x^4 + y^5 + xy^4 \in \mathbf{C}[x, y]$, if it is possible[4]!
- $f = x^3 + y^3 + z^3 \in CC[x, y, z]$.

4.3 Application 2: Logarithmic approximations to the ideal $Ann_{\mathcal{D}}(1/f^\alpha)$

The ring $R = \mathbf{C}[x_1, \dots, x_n]$ is a left A-module for the natural action defined as follows:

$$x_i \bullet f = x_i f,\ \partial_i \bullet f = \frac{\partial f}{\partial x_i}$$

for any $f \in R$. In fact, R is isomorphic, as an A-module, to the quotient of A by the left ideal generated by $\partial_1, \dots, \partial_n$.

Let us consider $f \in R$. The localization ring R_f (i.e. the ring of rational functions with poles along f) is the ring of quotients

$$R_f = \{\frac{g}{f^m} \,|\, g \in R,\ m \in \mathbf{N}\}.$$

R_f is a R-module and a left A-module in a natural way: the action $\partial_i \bullet \frac{g}{f^m}$ is just defined as the partial derivative of a rational function. Of course R_f is not a finitely generated R-module.

One of the main results in $\mathcal{D}$-module theory is the following theorem (see [6] or [7]):

Theorem 4.3.1 *Given any $f \in R$, the left A-module R_f is finitely generated. In fact, there exists a positive integer number α such that R_f is the left A-module generated by the rational function $\frac{1}{f^\alpha}$.*

[4] A challenge for any system.

The left A-module generated by $\frac{1}{f^k}$ is just the set

$$A\frac{1}{f^k} = \{P \bullet \frac{1}{f^k},\, P \in A\} \subset R_f.$$

The main ingredient in the proof of the last theorem is the existence of the called b-function $b_f(s)$ which has the following property: if $-\alpha$ is the least integer root of $b_f(s)$ then

$$R_f = A\frac{1}{f^\alpha}.$$

Bernstein proved ([6]) that the dimension of the characteristic variety of R_f is n, so R_f is *holonomic*.

In computational D-modules theory a natural problem is the following:

Problem.- Given a polynomial $f \in R$:

a) Compute a positive integer number $-\alpha$ such that $R_f = A\frac{1}{f^\alpha}$ and
b) Compute a system of generators of the annihilator $Ann_A(1/f^\alpha)$, i.e. compute a presentation

$$R_f = \frac{A}{Ann_A(\frac{1}{f^\alpha})}.$$

It is well known that there are algorithms to answer both questions: the global b-function was treated in the last section and it is well known that $Ann_A(1/f^\alpha)$ is obtained from $Ann_{A[s]}(f^s)$ setting $s = -\alpha$. Unfortunately, in many cases the available implementations of these methods can not obtain the results due to the unmanageable size of the Gröbner bases computations needed by the algorithms. It is possible to build —in the context of the so called *logarithmic* D-modules— some natural approximations of $Ann_A(1/f^\alpha)$.

Definition 4.3.2 *Let f be a polynomial in R. A derivation δ is called* logarithmic *for f, $f \in Der(\log f)$, if $\delta(f) = m \cdot f$ for some $m \in R$.*

Given an element $\delta = a_1\partial_1 + \cdots + a_n\partial_n \in Der(\log f)$, such that $\delta(f) = mf$ it is clear that $\delta + \alpha \cdot m$ annihilates $1/f^\alpha$. So, if $-\alpha$ is known to be the least integer root of $b_f(s)$, a natural approximation to $Ann_A(1/f^\alpha)$ is the ideal

$$\widetilde{I}^{\log f, -\alpha}) = \langle \delta + \alpha \cdot m \text{ such that } \delta(f) = m \cdot f\rangle.$$

Remark 4.3.3 *Note that, if the b-function or $-\alpha$ are unknown*[5] *it is far from being clear which is the correct logarithmic approximation!*

The point here is that $Der(\log f)$ and $\widetilde{I}^{\log f, -\alpha}$ are computable calculating syzygies among f and its derivatives:

$$(a_0, a_1, \ldots, a_n) \in Syz(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \Longleftrightarrow (\sum_{i=1}^{n} a_i\partial_i - \alpha a_0) \bullet (1/f^\alpha) = 0.$$

[5]There are many results about the roots of the b-functions for special cases and sometimes it is known the least integer root independently of the expression of b_f. It is the case of the plane curves, for example, for which -1 is the least integer root (Varchenko).

A very nice open problem[6] is:

Problem.- Given a polynomial $f \in R$, is $Ann_A(1/f^\alpha) \stackrel{?}{=} \widetilde{I}^{\log f, -\alpha}$?.

Exercise.- Check that $Ann_{A_2(\mathbf{C})}(1/f) = \widetilde{I}^{\log f, -1}$ for $f = x^2 + y^3$.

Exercise.- Calculate $\widetilde{I}^{\log f, -1}$ for $f = x^4 + y^5 + xy^4 \in \mathbf{C}[x, y]$. Prove that $Ann_{A_2(\mathbf{C})}(1/f) \neq \widetilde{I}^{\log f, -1}$ using the (commutative) calculation of the elements of $Ann_{A_2(\mathbf{C})}(1/f)$ of total degree at most 2 in the derivatives.

Exercise.- Do the ideals $Ann_{A_3(\mathbf{C})}(1/f)$ and $\widetilde{I}^{\log f, -2}$ coincide for $f = x^3 + y^3 + z^3 \in \mathbf{C}[x, y, z]$?

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