SIMPLICIAL COMPLEXES, GENERIC INITIAL IDEALS AND COMBINATORICS

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ABSTRACT. The results that will be presented in this talk were motivated by the technique of algebraic shifting, an operation on simplicial complexes introduced by Gil Kalai that converts a given simplicial complex into a combinatorially simpler (shifted) complex while preserving several invariants. There are two forms of shifting: symmetric shifting in the polynomial ring and exterior shifting in the exterior algebra. Both involve computing the reverse lex generic initial ideal (gin) of the Stanley-Reisner ideal of the input complex.

We introduce a new set of invariants for a homogeneous polynomial ideal called its symmetric iterated Betti numbers. In the context of shifting, these numbers have a combinatorial interpretation and contain among them the extremal Betti numbers in a minimal free resolution of the Stanley-Reisner ideal of the input complex.

This is joint work with Eric Babson and Isabella Novik at the University of Washington in Seattle. These notes are a shortened version of the paper [6].

1. INTRODUCTION AND THE MAIN RESULTS

The goal of this talk is to define and interpret a set of invariants of a homogeneous ideal in a polynomial ring, called the *symmetric iterated Betti numbers* of the ideal. These invariants were introduced in [6]. Large parts of these notes are taken directly from this paper which we refer to for details and complete proofs.

The notion of symmetric iterated Betti numbers for a polynomial ideal was motivated by *algebraic shifting*, a technique introduced by Gil Kalai in the eighties to study simplicial complexes. Let Γ be a simplicial complex on the vertex set $[n] := \{1, \ldots, n\}$. In [5] and [15], Kalai introduced two versions of algebraic shifting which given Γ, provides new simplicial complexes with the same vertex set. We denote these versions by $\Delta(\Gamma)$ for the *symmetric shifting* of Γ (see Definition 2.1) and by $\Delta^e(\Gamma)$ for the *exterior shifting* of Γ (see Definition 2.2). For both of these operations it is known that:

- (P1) $\Delta^{(e)}(\Gamma)$ is *shifted*, that is, for every $F \in \Delta^{(e)}(\Gamma)$, if $j < i \in F$, then $(F \setminus \{i\}) \cup \{j\} \in \Delta^{(e)}(\Gamma)$.
- (P2) If Γ is shifted, then $\Delta^{(e)}(\Gamma) = \Gamma$.
- (P3) Γ and $\Delta^{(e)}(\Gamma)$ have the same f-vector, that is, they have the same number of i-dimensional faces for every i.
- (P4) If Γ' is a subcomplex of Γ, then $\Delta^{(e)}(\Gamma') \subset \Delta^{(e)}(\Gamma)$.

Both versions were studied extensively from the algebraic point of view in a series of recent papers by Aramova, Herzog, Hibi and others (surveyed in [12]).

Consider the polynomial ring $S = \mathbf{k}[y_1, \ldots, y_n]$ where $\mathbf{k}$ is a field of characteristic zero. Let $\mathbb{N}$ denote the set of non-negative integers. If $A \subseteq [n]$ then write $y^A =$

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$\prod_{a\in A} y_a$. Denote by $\mathbb{N}^{[n]}$ the monomials of S by identifying a function $f:[n]\to\mathbb{N}$ in $\mathbb{N}^{[n]}$ with the monomial $\prod_{i\in[n]} y_i^{f(i)}$ and consider $\mathbb{N}^{[n]}$ as a multiplicative monoid. Thus $\{0,1\}^{[n]} = y^{2^{[n]}}$ is the set of squarefree monomials. If $\Gamma \subseteq 2^{[n]}$ is a simplicial complex then the *Stanley-Reisner* ideal of Γ [18, Def. II.1.1] is the squarefree monomial ideal

$$I_\Gamma := \langle y^{2^{[n]}-\Gamma}\rangle \subset S.$$

In other words, I_Γ is generated by all squarefree monomials in S whose supports are the minimal non-faces of Γ.

The (bi-graded) *Betti numbers* of a homogeneous ideal $I\subset S$ are the invariants $\beta_{i,j}(I)$ that appear in the minimal free resolution of I as an S-module.

$$\to \dots \bigoplus_j S(-j)^{\beta_{i,j}(I)} \to \dots \to \bigoplus_j S(-j)^{\beta_{1,j}(I)} \to \bigoplus_j S(-j)^{\beta_{0,j}(I)} \to I \to 0$$

Here $S(-j)$ denotes S with grading shifted by j. We say that $\beta_{i,i+j}(I)$, is *extremal* if $0\neq \beta_{i,i+j}(I) = \sum_{i'\geq i, j'\geq j}\beta_{i',i'+j'}(I)$. (This is equivalent to having $0\neq\beta_{i,i+j}(I)$ and $0=\beta_{i',i'+j'}(I)$ for every $i'\geq i$ and $j'\geq j$, $(i',j')\neq(i,j)$.) This terminology comes from the Betti diagram of I output by the computer algebra program *Macaulay 2* [11]. The extremal Betti numbers are the entries in the south-east corners of this Betti diagram.

Since $\Delta^{(e)}(\Gamma)$ are shifted complexes, their combinatorial structures are simpler than that of Γ. Nonetheless, $\Delta^{(e)}$ preserve many combinatorial and topological properties.

1. $\Delta^{(e)}$ preserve topological Betti numbers: (see [5, Thm. 3.1], [1, Prop. 8.3] for exterior shifting and [12, Cor. 8.25] for symmetric shifting). Moreover, exterior algebraic shifting preserves the *exterior iterated Betti numbers* of a simplicial complex. (There are two versions of exterior iterated Betti numbers — one due to Kalai [16, Cor. 3.4] and another due to Duval and Rose [8]. Both sets of numbers are preserved under exterior shifting.)
2. $\Delta^{(e)}$ preserve Cohen-Macaulayness: a simplicial complex Γ is Cohen-Macaulay if and only if $\Delta^{(e)}(\Gamma)$ is Cohen-Macaulay, which happens if and only if $\Delta^{(e)}(\Gamma)$ is pure (see [16, Thm. 5.3], [1, Prop. 8.4] for exterior shifting and [15, Thm. 6.4] for symmetric shifting).
3. $\Delta^{(e)}$ preserve extremal Betti numbers: $\beta_{i,i+j}(I_\Gamma)$ is an extremal Betti number of I_Γ if and only if $\beta_{i,i+j}(I_{\Delta^{(e)}(\Gamma)})$ is extremal for $I_{\Delta^{(e)}(\Gamma)}$, in which case $\beta_{i,i+j}(I_\Gamma) = \beta_{i,i+j}(I_{\Delta^{(e)}(\Gamma)})$ (see [3] for symmetric shifting and [1, Thm. 9.7] for both versions.)

Property 3 is a far-reaching generalization of Property 2, while Property 1 played a crucial role in Kalai's proof of Property 2 for exterior shifting. This suggests that there might be a connection between the iterated Betti numbers of a simplicial complex Γ on the one hand and the extremal Betti numbers of the ideal I_Γ on the other. This is one of the connections we will establish.

Recall that if M is an S-module, N is a submodule and I is an ideal in S then $(N:I^\infty)_M = \{m\in M \mid \text{for some } r\in\mathbb{N}, I^r m\subseteq N\}$ and if $I=\langle f\rangle$ it is typical to write $(N:\langle f\rangle^\infty) = (N:f^\infty)$. For an S-module M, the 0-th local cohomology of M with respect to the irrelevant ideal $\mathbf{m} = S_+ = \langle y_1,\dots,y_n\rangle$ is defined as

$$H^0(M) = \{m\in M : \mathbf{m}^k\cdot m = 0 \text{ for some } k\} = (0:\mathbf{m}^\infty)_M.$$

In particular $H^0(M)$ is graded when M is graded.

Consider the action of $GL(S_1)$ on S and choose $u \in GL(S_1)$ to be generic. Denote by $\mathbf{m} = \langle S_1 \rangle$ the irrelevant ideal of S. If I is a homogeneous ideal in S then write $J_0(I) = uI$ and $J_i(I) = y_i S + (J_{i-1}(I) : \mathbf{m}^\infty)$. We now come to the central definition.

Definition 1.1. The symmetric iterated Betti numbers of a homogeneous ideal I in S are

$$b_{i,r}(I) := \dim H^0(S/J_i(I))_r \quad \text{for } 0 \le i, r \le n,$$

where $H^0(-)_r$ stands for the r-th component of the 0-th local cohomology with respect to the irrelevant ideal $\mathbf{m}$.

If Γ is a simplicial complex with vertex set $[n]$, define the symmetric iterated Betti numbers of Γ to be $b_{i,r}(\Gamma) := b_{i,r}(I_\Gamma)$, $0 \le i, r \le n$.

Our first result gives a combinatorial interpretation of the symmetric iterated Betti numbers of a simplicial complex Γ and shows that they are invariant under symmetric algebraic shifting. Let $\max(\Gamma)$ denote the set of facets (maximal faces) of Γ. Write $\dim(\Gamma) = \max\{|F| - 1 : F \in \Gamma\}$.

Theorem 4.1. *Let Γ be a simplicial complex. Then*

$$b_{i,r}(\Gamma) = \begin{cases} |\{F \in \max(\Delta(\Gamma)) : |F| = i,\ [i-r] \subseteq F,\ i-r+1 \notin F\}| & \text{if } r \le i \\ 0 & \text{otherwise.} \end{cases}$$

In particular, since $\Delta(\Delta(\Gamma)) = \Delta(\Gamma)$, it follows that the symmetric iterated Betti numbers of Γ are invariant under symmetric shifting.

Theorem 4.1 implies that $b_{i,r}(\Gamma) = 0$ unless $0 \le r \le i \le \dim(\Gamma)+1$. The exterior iterated Betti numbers of Γ, $b^e_{i,r}(\Gamma)$, defined by Duval and Rose have precisely the same combinatorial formula (up to a slight change in indices), except that in their definition, one replaces $\Delta(\Gamma)$ by $\Delta^e(\Gamma)$ [8, Thm. 4.1].

The extremal Betti numbers of an ideal $I = I_\Gamma$ are the extremal iterated Betti numbers (symmetric or exterior) of the simplicial complex Γ in the following sense.

Theorem 4.5. *Let Γ be a simplicial complex. The extremal Betti numbers of I_Γ form a subset of the symmetric as well as of the exterior iterated Betti numbers of Γ. More precisely, $\beta_{j-1,i+j}(I_\Gamma)$ is an extremal Betti number of I_Γ if and only if*

$$b^{(e)}_{n-j',i'}(\Gamma) = 0 \quad \forall (i',j') \ne (i,j),\ i' \ge i,\ j' \ge j,\ \text{and } b^{(e)}_{n-j,i}(\Gamma) \ne 0.$$

In such a case $\beta_{j-1,i+j}(I_\Gamma) = b_{n-j,i}(\Gamma) = b^e_{n-j,i}(\Gamma)$.

Let $\mathrm{Gin}\,(I)$ denote the reverse lexicographic *generic initial ideal* of a homogeneous ideal I in S with variables ordered as $y_n \succ y_{n-1} \succ \cdots \succ y_1$. It follows from [3, Cor. 1.7] that the symmetric iterated Betti numbers of I coincide with those of $\mathrm{Gin}\,(I)$. We provide an alternate proof of this fact in [6]. Our third result (Theorem 4.13) interprets the symmetric iterated Betti numbers $b_{i,r}(I)$ in terms of the associated primes of $\mathrm{Gin}\,(I)$.

These notes are organized as follows. In Section 2 we recall the basics of algebraic shifting. Section 3 defines and interprets certain monomial sets that are at the root of all our proofs. In Section 4 we explain (without complete proofs) the theorems stated above.

2. ALGEBRAIC SHIFTING

In this section we recall the basics of symmetric and exterior algebraic shifting. For further details and open questions see the survey articles by Herzog [12] and Kalai [17].

Let $\mathbb{N}^\sigma$ denote the set of all finite degree monomials in the variables y_i with $i \in \sigma$ and $\mathbb{N}^\sigma_r$ denote the set of elements of degree r in $\mathbb{N}^\sigma$. In particular, if $[n] = [1, n] = \{1, \ldots, n\}$ then $\mathbb{N}^{[n]}$ is the set of all monomials in S and $\{0, 1\}^\sigma$ is the set of all square free finite degree monomials in $\mathbb{N}^\sigma$. In this paper we fix the reverse lexicographic order $\succ$ on $\mathbb{N}^\mathbb{Z}$ with $y_i \succ y_{i-1}$ for all $i \in \mathbb{Z}$ extending the partial ordering by degree. We also define the square free map $\Phi : \mathbb{N}^\mathbb{Z} \to \{0, 1\}^\mathbb{Z}$ to be the unique degree and order preserving bijection for which $\Phi(y_0^n) = \prod_{-n<i\leq 0} y_i$. Thus for example $\Phi(y_4 y_6^3 y_7) = y_0 y_3 y_4 y_5 y_7$.

For each homogeneous ideal $I \subset S$ there exists a Zariski open set $U(I) \subset GL(S_1)$ such that the ideal $\text{In}_\succ(uI)$, (the *initial ideal* of uI with respect to the monomial order $\succ$ on S), is independent of the choice of $u \in U(I)$. The ideal $\text{In}_\succ(uI)$ is called the *generic initial ideal* of I with respect to $\succ$ and is denoted by $\text{Gin}\,(I) = \text{Gin}_\succ(I)$ (see [9, Chapter 15]). If I is a homogeneous ideal in S then one way to explicitly and uniformly construct an element $\alpha \in U(I)$ is to consider the extension $\mathbf{K} = \mathbf{k}(\{\alpha_{i,j}\}_{i,j\in[n]})/\mathbf{k}$ and then for any ideal I in S the element

$$\alpha : S_\mathbf{K} = S \otimes_\mathbf{k} \mathbf{K} \to S_\mathbf{K} \quad \text{given by} \quad \alpha y_i = \sum_{j=1}^{n} \alpha_{i,j} y_j$$

is generic for $\mathbf{K}I$ as an ideal of $S_\mathbf{K}$.

For a homogeneous ideal I in S and a generic linear map $u \in U(I)$ define

$$B(I) = \{m \in \mathbb{N}^{[n]} \ : \ m \,\text{is not in the linear span of}\ \{n|m \succ n\} \cup uI\}.$$

Note that $B(I)$ is a basis of the vector space $M_0(I) = S/uI$ and hence $B(I) = \mathbb{N}^{[n]} - \text{Gin}\,(I)$, the set of standard monomials of $\text{Gin}\,(I)$.

Definition 2.1. The symmetric algebraic shifting of a simplicial complex $\Gamma \subseteq 2^{[n]}$ is $\Delta(\Gamma)$ where $y^{\Delta(\Gamma)} = \Phi(B(I_\Gamma)) \cap \mathbb{N}^{[1,\infty]} \subseteq \{0, 1\}^{[n]}$.

Note that this means that $I_{\Delta(\Gamma)} = \langle \Phi(\mathbb{N}^{[n]} - B(I_\Gamma)) \rangle$.

The fact that $\Delta(\Gamma)$ is a simplicial complex satisfying conditions (P1)–(P4) was proved in [15, Thm. 6.4], [2] by using certain properties of $B(I)$. We list some of them below.

- (B1) $B(I)$ is a basis of S/uI, as well as of $S/\text{Gin}\,(I)$.
- (B2) $B(I)$ is an order ideal — if $m \in B(I)$ and $m'|m$, then $m' \in B(I)$.
- (B3) $B(I)$ is shifted — if $j < i$ and $y_i m \in B(I)$ then $y_j m \in B(I)$.

(B1) was discussed above while (B2) follows from the fact that $\text{Gin}\,(I)$ is an ideal. (B3) is a consequence of the fact that generic initial ideals are Borel fixed [9, Theorem 15.20]. In characteristic 0, this is equivalent to $\text{Gin}\,(I)$ being *strongly stable* [9, Theorem 15.23], which means that if $j < i$ and $y_j m \in \text{Gin}\,(I)$ then $y_i m \in \text{Gin}\,(I)$. It turns out that if M is strongly stable then ΦM has the same Hilbert function as M. In fact the two ideals have the same Betti diagrams. However, if M is not strongly stable, Φ need not preserve the Hilbert function.

In the case when $I = I_\Gamma$, $B(I_\Gamma)$ has another fundamental property:

- (B4) If $m \in B(I_\Gamma) \cap \mathbb{N}_r^{[k,n]}$ and $r \geq k$ then $m\mathbb{N}^{\{k\}} \subseteq B(I_\Gamma)$ as well.

This is due to Kalai [15, Lemma 6.3] and implies that $y_1, \dots, y_n$ is an *almost regular* $M_0(I_\Gamma)$-sequence (a notion introduced by Aramova and Herzog [1]; it played a crucial role in their proof that extremal Betti numbers are preserved by algebraic shifting).

Let $E = \bigwedge(\mathbf{k}[y_1, \dots, y_n]_1) = \bigwedge S_1$ be the exterior algebra over the n-dimensional vector space S_1. A *monomial* in E is an expression of the form $m = y_{i_1} \wedge y_{i_2} \wedge \cdots \wedge y_{i_k}$, where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$; the set $\{i_1, i_2, \dots, i_k\}$ is called the support of m, and is denoted by $\mathrm{supp}\,(m)$. The *exterior Stanley-Reisner ideal* of a simplicial complex Γ on $[n]$ is

$$J_\Gamma := \langle m \in E \,:\, m \text{ is a monomial, } \mathrm{supp}\,(m) \notin \Gamma \rangle.$$

Definition 2.2. The exterior algebraic shifting of Γ, $\Delta^e(\Gamma)$, is the simplicial complex defined by $J_{\Delta^e(\Gamma)} := \mathrm{Gin}\,(J_\Gamma)$, where $\mathrm{Gin}\,(J_\Gamma)$ is the generic initial ideal of J_Γ with respect to the reverse lexicographic order with $y_n \succ y_{n-1} \succ \cdots \succ y_1$.

3. Special monomial subsets

In this section we identify and interpret certain subsets of monomials in the basis $B(I)$ of $M_0(I)$ that are at the root of the proofs of our main theorems.

Definition 3.1. Let I be a homogeneous ideal in S. For $i \in [0, n]$ define

$$\begin{aligned} A_i(I) &:= \left\{ \ m \in \mathbb{N}^{[i+1,n]} \ : \ m\mathbb{N}^{\{i\}} \subseteq B(I), \ \ m\mathbb{N}^{\{i+1\}} \nsubseteq B(I) \ \right\} \\ A_{i,r}(I) &:= A_i(I) \cap \mathbb{N}^{\mathbb{N}}_r. \end{aligned}$$

Several remarks are in order. Since $B(I)$ is shifted (B3), $m\mathbb{N}^{\{i\}} \subseteq B(I)$ iff $m\mathbb{N}^{[i]} \subseteq B(I)$. Since $B(I_\Gamma)$ satisfies (B4),

$$(1) \qquad A_{i,r}(I_\Gamma) = \emptyset \qquad \text{if } r > i \qquad \text{and hence} \qquad A_i(I_\Gamma) = \cup_{r=0}^{i} A_{i,r}(I_\Gamma).$$

Also if $m \in \mathbb{N}_r^{[r,n]}$ then $m \in B(I_\Gamma)$ iff $m\mathbb{N}^{[r]} \subseteq B(I_\Gamma)$. Hence

$$(2) \qquad A_{i,r}(I_\Gamma) = \left\{ m \in \mathbb{N}_r^{[i+1,n]} : y_i^{i-r} \cdot m \in B(I_\Gamma), \ y_{i+1}^{i-r+1} \cdot m \notin B(I_\Gamma) \right\}.$$

In [19], Sturmfels, Trung and Vogel introduced a decomposition of the standard monomials of an arbitrary monomial ideal M, called its *standard pair decomposition*, in order to study the multiplicities of associated primes and degrees of M. Theorem 4.13 interprets these quantities for the monomial ideals I_Γ, $I_{\Delta(\Gamma)}$, and $\mathrm{Gin}\,(I)$ in terms of symmetric iterated Betti numbers of the gin. This in turn relies on the fact that the sets of monomials $A_i(I)$ defined above index the *standard pairs* of $\mathrm{Gin}\,(I)$. For a monomial $m \in \mathbb{N}^{\mathbb{Z}}$, let $\mathrm{supp}\,(m) := \{i \,:\, y_i | m\} \subset \mathbb{Z}$ be called the *support* of m. Thus $\mathrm{supp}\, : \{0,1\}^\sigma \to 2^\sigma$ is a bijection.

Definition 3.2. [19] Let $M = \langle M \cap \mathbb{N}^{[n]} \rangle \subseteq S$ be a monomial ideal. A **standard monomial** of M is an element of $\mathbb{N}^{[n]} - M$. An **admissible pair** of M is a subset $m\mathbb{N}^\sigma \subseteq \mathbb{N}^{[n]} - M$ with $m \in \mathbb{N}^{[n]-\sigma}$ or equivalently if we take $\mathbb{Z}^\sigma$ to be Laurent monomials then an admissible pair is a subset $m\mathbb{Z}^\sigma \cap \mathbb{N}^{[n]}$ with $m\mathbb{Z}^\sigma \cap M = \emptyset$. A **standard pair** of M is a(n inclusion) maximal admissible pair.

Our starting point is the following relationship between the sets defined above and the standard pairs of $\mathrm{Gin}\,(I)$. For the proof of this lemma as well as all other proofs omitted from these notes, please see [6].

Lemma 3.3. *If $I \subseteq S$ is an ideal then the standard pairs of* $\mathrm{Gin}\,(I)$ *are* $\{a\mathbb{N}^{[i]} : a \in A_i(I)\}$. *(Here $[0] = \emptyset$.)*

Corollary 3.4. *If $m\mathbb{N}^{[i]}$ is a standard pair of* $\mathrm{Gin}\,(I_\Gamma)$ *then the degree of m is at most i.*

The standard pairs of monomial ideals of moderate size can be computed using the computer algebra package *Macaulay 2* [11] (see the chapter *Monomial Ideals* in [10] for details). This gives a method for computing the sets $A_i(I)$ for small examples — see Example 3.9 below.

Example 3.5. Consider the ideal $I = \langle z^6 - 5z^4y^2, z^3yx^3 - 3xyz^5, y^2z^2\rangle \subset k[x, y, z]$. Under the reverse lexicographic order $\succ$ with $z \succ y \succ x$,

$$\mathrm{Gin}\,(I) = \langle z^4, y^3z^3, y^5z^2, xy^4z^2, x^3y^2z^3, x^5yz^3\rangle.$$

The standard pairs of $\mathrm{Gin}\,(I)$ are:

- $\mathbb{N}^{\{1,2\}}, z\mathbb{N}^{\{1,2\}}$,
- $z^2\mathbb{N}^{\{1\}}, yz^2\mathbb{N}^{\{1\}}, y^2z^2\mathbb{N}^{\{1\}}, y^3z^2\mathbb{N}^{\{1\}}, z^3\mathbb{N}^{\{1\}}$,
- $y^4z^2\mathbb{N}^{\emptyset}, yz^3\mathbb{N}^{\emptyset}, y^2z^3\mathbb{N}^{\emptyset}, xy^2z^3\mathbb{N}^{\emptyset}, x^2y^2z^3\mathbb{N}^{\emptyset}, xyz^3\mathbb{N}^{\emptyset}, x^2yz^3\mathbb{N}^{\emptyset}$, $x^3yz^3\mathbb{N}^{\emptyset}, x^4yz^3\mathbb{N}^{\emptyset}$.

Figure 1 shows the decomposition of the standard monomials of $\mathrm{Gin}\,(I)$ given by its standard pairs. The generators of $\mathrm{Gin}\,(I)$ are the labeled black dots and the standard pair $y^m\mathbb{N}^\sigma$ is depicted by the cone $m + \mathbb{R}^\sigma_{\geq 0}$.

In the case when $I = I_\Gamma$ there is another interpretation of the monomials in $A_i(I_\Gamma)$ that relates them to the shifted complex $\Delta(\Gamma)$, and is useful for the proofs of Theorems 4.1 and 4.13.

Lemma 3.6. *There is a bijection between the sets*

$$A_{i,r}(I_\Gamma) \quad \textit{and} \quad \{F \in \max(\Delta(\Gamma)) : |F| = i,\ [i-r] \subseteq F,\ i-r+1 \notin F\}$$

given by Φ with $A_{i,r}(I_\Gamma) \ni m \mapsto [i-r] \cup \mathrm{supp}\,(\Phi(m)) = \mathrm{supp}\,(\Phi(my_i^{i-r}))$.

Corollary 3.7. *The standard pairs of* $\mathrm{Gin}\,(I_\Gamma)$ *are in bijection with the facets of* $\Delta(\Gamma)$*: $m\mathbb{N}^{[i]}$ is a standard pair of* $\mathrm{Gin}\,(I_\Gamma)$ *if and only if* $[i-r] \cup \mathrm{supp}\,(\Phi(m))$ *is a facet of $\Delta(\Gamma)$ of size i.*

Lemma 3.6 along with the fact that $b_{i,r}(I) = |A_{i,r}(I)|$ for all $i, r \in [0, n]$, proves Theorem 4.1. (Hence, in particular, it follows from Theorem 4.1 that $A_i(I_\Gamma) = \emptyset$ for all $i > \dim(\Gamma) + 1$.) Thus the sets $A_{i,r}(I_\Gamma)$ and their cardinalities $b_{i,r}(\Gamma)$ carry important information about Γ, and we record them in the following *triangles*.

Definition 3.8. The b-triangle and monomial b-triangle of a simplicial complex Γ are the lower triangular matrices whose respective (i, r)-th entries are $b_{i,r}(\Gamma)$ and $A_{i,r}(I_\Gamma)$ for $0 \leq i \leq r \leq \dim(\Gamma) + 1$.

Example 3.9. Let Γ be the simplicial complex whose facets are

$$\begin{aligned}\max(\Gamma) = \{&\{1,2,4\}, \{1,2,6\}, \{1,3,4\}, \{1,3,7\}, \{1,5,6\}, \{1,5,7\}, \{2,3,5\},\\ &\{2,3,7\}, \{2,4,5\}, \{2,6,7\}, \{3,4,6\}, \{3,5,6\}, \{4,5,7\}, \{4,6,7\}\}.\end{aligned}$$

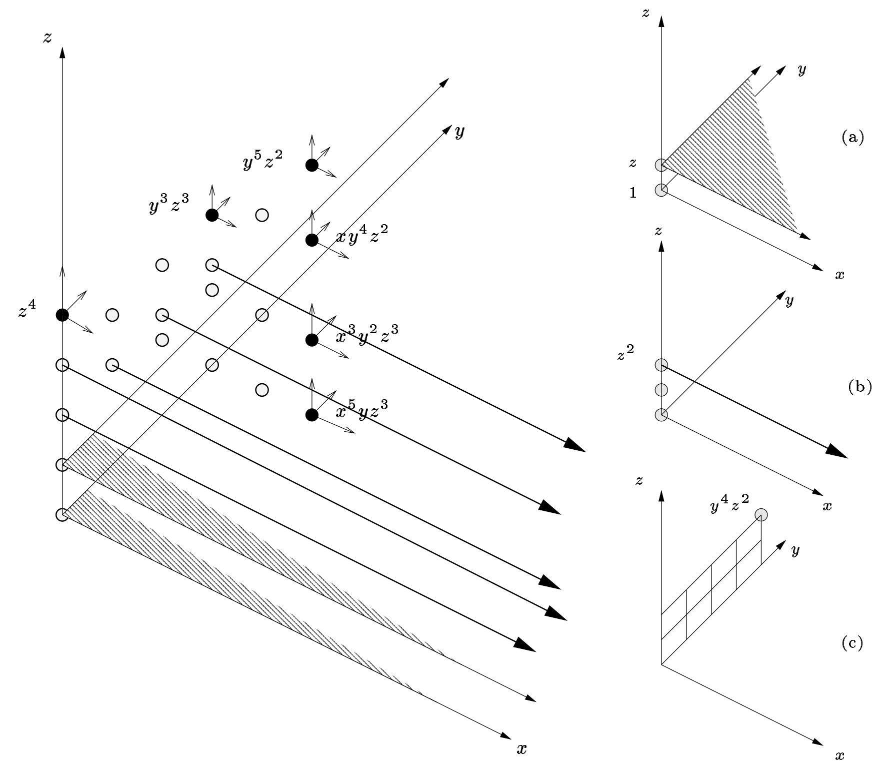

FIGURE 1. The standard pair decomposition of $\text{Gin}(I)$ where for instance, the standard pairs $z\mathbb{N}^{\{1,2\}}$, $z^2\mathbb{N}^{\{1\}}$ and $y^4z^2\mathbb{N}^{\emptyset}$ are shown separately in (a), (b) and (c) respectively.

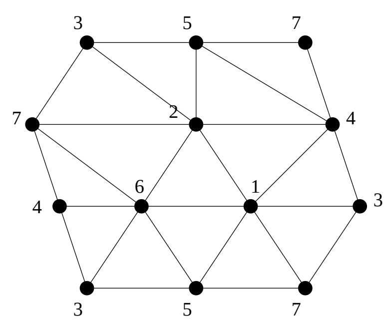

FIGURE 2. The simplicial complex Γ in Example 3.9. Here parallel boundary regions are identified.

Then the Stanley-Reisner ideal of Γ in the ring $S := \mathbf{k}[a, b, c, d, e, f, g]$ is:

$$I_\Gamma = \langle efg, cfg, afg, ceg, beg, cdg, bdg, adg, abg, def, bef, bdf, adf, bcf, acf, cde, ade, ace, abe, bcd, abc\rangle.$$

Under the reverse lexicographic order $\succ$ with $g \succ f \succ \cdots \succ b \succ a$,

$$\begin{aligned}\text{Gin}\,(I_\Gamma) \;&=\; \langle gf^2, f^3, f^2e, g^2f, gfe, fe^2, gfd, f^2d, fed, g^2e, ge^2, e^3,\\ &\quad ged, e^2d, fd^2, g^3, g^2d, gd^2, ed^2, g^2c, gfc, d^4\rangle.\end{aligned}$$

Applying the map Φ to the generators of Gin (I_Γ) we get

$$\begin{aligned}I_{\Delta(\Gamma)} \;&=\; \langle gea, gfa, ecb, fcb, gcb, edb, fdb, gdb, feb, geb, gfb, edc,\\ &\quad fdc, gdc, fec, gec, gfc, fed, ged, gfd, gfe, dcba\rangle,\end{aligned}$$

which shows that the shifted complex $\Delta(\Gamma)$ has facets:

$$\begin{aligned}\max(\Delta(\Gamma)) \;&=\; \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,2,7\},\\ &\quad \{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,3,7\},\{1,4,5\},\\ &\quad \{1,4,6\},\{1,4,7\},\{1,5,6\},\{2,3,4\},\{5,7\},\{6,7\}\}.\end{aligned}$$

The one skeleton of $\Delta(\Gamma)$ is (like that of Γ) the complete graph on it's 7 vertices. The triangles are obtained by coning 1 with all edges involving the vertices $2,3,4,5,6$ and 7 except for $\{5,7\}$ and $\{6,7\}$ and adding the triangle $\{2,3,4\}$. Thus all the triangles and the edges $\{5,7\}$ and $\{6,7\}$ are facets.

We compute the b-triangle and the monomial b-triangle of Γ by first computing the standard pairs of Gin (I_Γ) using Macaulay 2.

b-triangle of Γ					monomial b-triangle of Γ				
	0	1	2	3		0	1	2	3
0	0				0	$\emptyset$			
1	0	0			1	$\emptyset$	$\emptyset$		
2	0	0	2		2	$\emptyset$	$\emptyset$	$\{g^2, gf\}$	
3	1	4	8	1	3	$\{1\}$	$\{g,f,e,d\}$	$\{ge, gd, f^2, fe, fd, e^2, ed, d^2\}$	$\{d^3\}$

The standard pairs of I_Γ, Gin (I_Γ) and $I_{\Delta(\Gamma)}$ are shown in the following table. Columns 2,3, and 4 illustrate Lemma 3.6 and Corollary 3.7.

StdPairs(I_Γ) form: $\mathbb{N}^\sigma$	StdPairs(Gin (I_Γ)) form: $m\mathbb{N}^{[i]}$	$\Phi(m)$ $\to$ supp $(\Phi(m))$	StdPairs($I_{\Delta(\Gamma)}$) form: $\mathbb{N}^\sigma$
$\mathbb{N}^{\{4,6,7\}}$	$\mathbb{N}^{\{1,2,3\}}$	$1 \to \emptyset$	$\mathbb{N}^{\{1,2,3\}}$
$\mathbb{N}^{\{2,6,7\}}$	$g\mathbb{N}^{\{1,2,3\}}$	$g \to \{7\}$	$\mathbb{N}^{\{1,2,7\}}$
$\mathbb{N}^{\{4,5,7\}}$	$ge\mathbb{N}^{\{1,2,3\}}$	$gd \to \{7,4\}$	$\mathbb{N}^{\{1,4,7\}}$
$\mathbb{N}^{\{1,5,7\}}$	$gd\mathbb{N}^{\{1,2,3\}}$	$gc \to \{7,3\}$	$\mathbb{N}^{\{1,3,7\}}$
$\mathbb{N}^{\{2,3,7\}}$	$f\mathbb{N}^{\{1,2,3\}}$	$f \to \{6\}$	$\mathbb{N}^{\{1,2,6\}}$
$\mathbb{N}^{\{1,3,7\}}$	$f^2\mathbb{N}^{\{1,2,3\}}$	$fe \to \{6,5\}$	$\mathbb{N}^{\{1,5,6\}}$
$\mathbb{N}^{\{3,5,6\}}$	$fe\mathbb{N}^{\{1,2,3\}}$	$fd \to \{6,4\}$	$\mathbb{N}^{\{1,4,6\}}$
$\mathbb{N}^{\{1,5,6\}}$	$fd\mathbb{N}^{\{1,2,3\}}$	$fc \to \{6,3\}$	$\mathbb{N}^{\{1,3,6\}}$
$\mathbb{N}^{\{3,4,6\}}$	$e\mathbb{N}^{\{1,2,3\}}$	$e \to \{5\}$	$\mathbb{N}^{\{1,2,5\}}$
$\mathbb{N}^{\{1,2,6\}}$	$e^2\mathbb{N}^{\{1,2,3\}}$	$ed \to \{5,4\}$	$\mathbb{N}^{\{1,4,5\}}$
$\mathbb{N}^{\{2,4,5\}}$	$ed\mathbb{N}^{\{1,2,3\}}$	$ec \to \{5,3\}$	$\mathbb{N}^{\{1,3,5\}}$
$\mathbb{N}^{\{2,3,5\}}$	$d\mathbb{N}^{\{1,2,3\}}$	$d \to \{4\}$	$\mathbb{N}^{\{1,2,4\}}$
$\mathbb{N}^{\{1,3,4\}}$	$d^2\mathbb{N}^{\{1,2,3\}}$	$dc \to \{4,3\}$	$\mathbb{N}^{\{1,3,4\}}$
$\mathbb{N}^{\{1,2,4\}}$	$d^3\mathbb{N}^{\{1,2,3\}}$	$dcb \to \{4,3,2\}$	$\mathbb{N}^{\{2,3,4\}}$
	$g^2\mathbb{N}^{\{1,2\}}$	$gf \to \{7,6\}$	$\mathbb{N}^{\{6,7\}}$
	$gf\mathbb{N}^{\{1,2\}}$	$ge \to \{7,5\}$	$\mathbb{N}^{\{5,7\}}$

4. The main theorems

Our first theorem provides a simple combinatorial formula for the symmetric iterated Betti numbers of a simplicial complex.

Theorem 4.1. *For a simplicial complex* Γ

$$b_{i,r}(\Gamma) = \begin{cases} |\{F \in \max(\Delta(\Gamma)) : |F| = i, [i-r] \subseteq F, i-r+1 \notin F\}| & \text{if } r \le i \\ 0 & \text{otherwise} \end{cases}$$

The symmetric iterated Betti numbers $b_{i,r}(\Gamma)$ were defined as the dimensions of the vector spaces $H^0(M_i(I_\Gamma))_r$, where for a homogeneous ideal I in S and a generic linear map $u \in U(I)$, $M_0(I) = S/uI$ and $M_i(I) = M_{i-1}(I)/(y_i M_{i-1}(I) + H^0(M_{i-1}(I)))$ for $1 \le i \le n$. Thus at step i we "peel off" the i-th variable. This is similar to the "deconing" of the shifted complex $\Delta(\Gamma)$ used in the definition of the exterior iterated Betti numbers of Γ by Duval and Rose [8].

In view of Lemma 3.6, it suffices to prove the following lemma in order to prove Theorem 4.1. The proof is rather technical and can be found in [6].

Lemma 4.2. $|A_{i,r}(I)| = \dim H^0(M_i(I))_r$ *for all* $i, r \ge 0$.

Let Γ be a simplicial complex on the vertex set $[n]$. The *Alexander dual* of Γ is the simplicial complex

$$\Gamma^* = \{F \subseteq [n] : [n] \setminus F \notin \Gamma\}.$$

The next two results (both due to Bayer, Charalambous and Popescu [3], see [1] also for the second theorem) provide connections between the extremal Betti numbers of the Stanley-Reisner ideals of Γ and Γ^*, and the shifted complex $\Delta(\Gamma)$.

Theorem 4.3. *Let* Γ *be a simplicial complex and* Γ^* *be its Alexander dual. The Stanley-Reisner ideals* I_Γ *and* I_{Γ^*} *have the same extremal Betti numbers. More precisely,* $\beta_{i,i+j}(I_{\Gamma^*})$ *is extremal if and only if* $\beta_{j-1,i+j}(I_\Gamma)$ *is extremal. Also, in such a case* $\beta_{i,i+j}(I_{\Gamma^*}) = \beta_{j-1,i+j}(I_\Gamma)$.

Theorem 4.4. *Extremal Betti numbers are preserved by algebraic shifting: for a simplicial complex* Γ, $\beta_{i,i+j}(I_\Gamma)$ *is extremal if and only if* $\beta_{i,i+j}(I_{\Delta(\Gamma)})$ *is extremal. Moreover, in such a case* $\beta_{i,i+j}(I_\Gamma) = \beta_{i,i+j}(I_{\Delta(\Gamma)})$.

Our second theorem is the following.

Theorem 4.5. *The extremal Betti numbers of* I_Γ *are contained among the symmetric iterated Betti numbers of* Γ. *They are precisely the extremal entries in the* b*-triangle of* Γ: $\beta_{j-1,i+j}(I_\Gamma)$ *is an extremal Betti number of* I_Γ *if and only if*

$$b_{n-j',i'}(\Gamma) = 0 \quad \forall (i',j') \neq (i,j),\ i' \ge i,\ j' \ge j, \text{ and } b_{n-j,i}(\Gamma) \neq 0.$$

Moreover, in this case, $\beta_{j-1,i+j}(I_\Gamma) = b_{n-j,i}(\Gamma)$.

Example 3.9 continued: The minimal free resolution and Betti diagram of I_Γ (computed by *Macaulay 2*) are given below. Note that the entries in the southeast corners of the Betti diagram of I_Γ (the extremal Betti numbers of I_Γ) are precisely the entries in the north-east corners of the b-triangle of Γ from Section 3.

$$0 \to S^2 \to S^{15} \to S^{42} \to S^{49} \to S^{21} \to S \to 0$$

total:	1	21	49	42	15	2
0 :	1	.	.	.	.	.
1 :	.	.	.	.	.	.
2 :	.	21	49	42	14	2
3 :	.	.	.	.	1	.

The proof of Theorem 4.5 relies on the following lemma, which is a consequence of [13, Thm. 2.1(b)] (see also [14, Prop. 12]) and [7, Cor. 6.2]. A different proof can be found in [6].

Lemma 4.6. *The symmetric iterated Betti numbers of* Γ *are related to the graded Betti numbers of the Stanley-Reisner ideal* $I_{\Delta(\Gamma^*)}$ *as follows:*

$$\beta_{i,i+j}(I_{\Delta(\Gamma^*)}) = \sum_r \binom{n-r-j}{i} b_{n-j,n-r-j}(\Gamma).$$

It is well known and is easy to prove that $\Delta(\Gamma^*) = \Delta(\Gamma)^*$.

Proof of Theorem 4.5: The theorem is an easy consequence of Lemma 4.6. Indeed, since $\binom{n-r-j}{i}$ is positive for $r \leq n-i-j$ and is zero otherwise, it follows from the lemma that

$$\beta_{i',i'+j'}(I_{\Delta(\Gamma^*)}) = 0 \quad \text{iff} \quad b_{n-j',n-j'-r}(\Gamma) = 0 \text{ for all } r \leq n-i'-j'.$$

Thus,

$$\begin{gathered} \beta_{i,i+j}(I_{\Delta(\Gamma^*)}) \neq 0 \text{ is extremal} \quad \Longleftrightarrow \\ \beta_{i',i'+j'}(I_{\Delta(\Gamma^*)}) = 0 \text{ for all } i' \geq i,\ j' \geq j,\ (i,j) \neq (i',j') \quad \Longleftrightarrow \\ b_{n-j',i'}(\Gamma) = 0 \text{ for all } i' \geq i,\ j' \geq j,\ (i,j) \neq (i',j'). \end{gathered}$$

Moreover, if this is the case, then all except the first summand in

$$\beta_{i,i+j}(I_{\Delta(\Gamma^*)}) = \binom{i}{i} b_{n-j,i}(\Gamma) + \sum_{r<n-i-j} \binom{n-r-j}{i} b_{n-j,n-r-j}(\Gamma)$$

vanish, implying that $\beta_{i,i+j}(I_{\Delta(\Gamma^*)}) = b_{n-j,i}(\Gamma)$. The result then follows from Theorems 4.3 and 4.4. □

Using [1, Cor . 1.2] and certain properties of sets $A_i(I)$ one can also prove the following more general result.

Theorem 4.7. *Let* I *be a homogeneous ideal. The extremal Betti numbers of* I *form a subset of the symmetric iterated Betti numbers of* I*. More precisely,* $\beta_{j-1,i+j}(I)$ *is an extremal Betti number of* I *if and only if*

$$b_{n-j',i'}(I) = 0 \quad \forall (i',j') \neq (i,j),\ i' \geq i,\ j' \geq j, \text{ and } b_{n-j,i}(I) \neq 0.$$

Moreover, in this case, $\beta_{j-1,i+j}(I) = b_{n-j,i}(I)$.

We now turn to the third theorem. The associated primes of a homogeneous ideal $I \subset S$ with a primary decomposition $I = Q_1 \cap Q_2 \cap \cdots \cap Q_t$ are the prime ideals $P_i := \sqrt{Q_i}$, $i = 1, \ldots, t$, where $\sqrt{Q_i}$ denotes the radical of Q_i. The set of associated primes of I, customarily denoted as $\mathrm{Ass}\,(I)$, is independent of the primary decomposition of I. The minimal elements of $\mathrm{Ass}\,(I)$ with respect to inclusion are called the *minimal primes* of I. We denote the set of minimal primes of I as $\mathrm{Min}\,(I)$. Recall that the irreducible (isolated) components of $V(I)$, the variety of I in $\mathbf{k}^n$, are the varieties $V(P)$ for $P \in \mathrm{Min}\,(I)$. Let $Z_i := V(P_i)$ be the variety of P_i in $\mathbf{k}^n$. The finite invariant $\deg(Z_i)$, called the *degree* of Z_i, is the cardinality of $Z_i \cap L$ for almost all linear subspaces L of dimension equal to the codimension of Z_i.

Definition 4.8. [4], [19]

(1) If P is a homogeneous prime ideal in S then the **multiplicity** of P (with respect to I), denoted as $\mathrm{mult}_I(P)$ is the length of the largest ideal of finite length in the ring S_P/IS_P.
(2) The **degree** of I, $\deg(I) := \sum_{\{\dim(Z_i)=\dim(I)\}} \mathrm{mult}_I(P_i)\deg(Z_i)$.
(3) The **geometric degree** of I,

$$\mathrm{geomdeg}\,(I) := \sum_{\{P_i \in \mathrm{Min}\,(I)\}} \mathrm{mult}_I(P_i)\deg(Z_i).$$

(4) The **arithmetic degree** of I,

$$\mathrm{arithdeg}\,(I) := \sum_{\{P_i \in Ass(I)\}} \mathrm{mult}_I(P_i)\deg(Z_i).$$

The invariant $\mathrm{mult}_I(P) > 0$ if and only if $P \in \mathrm{Ass}\,(I)$. Our main goal in this section is to prove Theorem 4.13. We first specialize Definition 4.8 to monomial ideals. If M is a monomial ideal, then every associated prime of M is of the form $P_\sigma := \langle y_j : j \notin \sigma \rangle$ for some set $\sigma \subseteq [n]$. Hence $V(P_\sigma)$ is the $|\sigma|$-dimensional linear subspace spanned by $\{e_j : j \in \sigma\}$ and $\deg(V(P_\sigma)) = 1$. The three degrees of M from Definition 4.8 are therefore appropriate sums of multiplicities of ideals in $\mathrm{Ass}\,(M)$ with respect to M.

For a monomial ideal M the multiplicities of associated primes as well as all the degrees referred to in Definition 4.8 can be read off from the standard pairs of M (see Definition 3.2) as shown in the following lemma. The statements in this lemma are either stated or can be derived easily from the results in [19].

Lemma 4.9. *Let M be a monomial ideal. Then,*

(1) *the set of standard pairs of M is well defined,*
(2) $*\mathbb{N}^\sigma$ *is a standard pair of M if and only if $P_\sigma \in \mathrm{Ass}\,(M)$,*
(3) $\mathbb{N}^\sigma$ *is a standard pair of M if and only if $P_\sigma \in \mathrm{Min}\,(M)$,*
(4) *the dimension of M is the maximal size of a set σ such that $*\mathbb{N}^\sigma$ is a standard pair of M,*
(5) *if $P_\sigma \in \mathrm{Ass}\,(M)$, then $\mathrm{mult}_M(P_\sigma)$ is the number of standard pairs of M of the form $*\mathbb{N}^\sigma$ and*
(6) (a) *$deg(M)$ is the number of standard pairs $*\mathbb{N}^\sigma$ of M such that $|\sigma| = dim(M)$,*
    (b) *$geomdeg(M)$ is the number of standard pairs $*\mathbb{N}^\sigma$ of M such that $\mathbb{N}^\sigma$ is a standard pair of M and*
    (c) *$arithdeg(M)$ is the total number of standard pairs of M.*

Lemma 3.3 showed that $m\mathbb{N}^\sigma$ is a standard pair of $\mathrm{Gin}\,(I)$ if and only if $\sigma = [i]$ and $m \in A_i(I)$ for some $0 \le i \le n$. Combining this fact with Lemma 4.9 we obtain the following.

Corollary 4.10. *(see also* [9, Corollary 15.25]*)*

(i) *$P_{[d]}$, $d = dim(I)$, is the unique minimal prime of $\mathrm{Gin}\,(I)$ (if $I = I_\Gamma$ then $d = \dim\Gamma + 1$), and*

(ii) *all embedded primes of* $\mathrm{Gin}\,(I)$ *are of the form* $P_{[k]}$ *for some* $k < d$.

Thus the submonoids in the standard pairs of $\mathrm{Gin}\,(I)$ are initial intervals of $[n]$ while the cosets can be complicated. On the other hand, for the square free monomial ideals I_Γ and $I_{\Delta(\Gamma)}$, the cosets of the standard pairs are trivial and the submonoids determine the ideals (cf. Example 3.9).

Corollary 4.11. *If* Γ *is a simplicial complex then* $I_\Gamma = \bigcap_{\sigma \in \max(\Gamma)} P_\sigma$ *is the irredundant prime decomposition of* I_Γ. *In particular,* I_Γ *has no embedded primes and its standard pairs are* $\{\mathbb{N}^\sigma \,:\, \sigma \in \max(\Gamma)\}$.

By Corollary 3.7, $m\mathbb{N}^{[i]}$ is a standard pair of $\mathrm{Gin}\,(I_\Gamma)$ if and only if $[i-r] \cup \mathrm{supp}\,(\Phi(m))$ is a facet of $\Delta(\Gamma)$ of size i. Combining this fact with Corollary 4.11 we get the following bijection as well.

Corollary 4.12. *There is a bijection between the standard pairs of* $\mathrm{Gin}\,(I_\Gamma)$ *and those of* $I_{\Delta(\Gamma)}$ *given by:* $m\mathbb{N}^{[i]}$ *is a standard pair of* $\mathrm{Gin}\,(I_\Gamma)$ *with* $\deg(m) = r$ *if and only if* $\mathbb{N}^{[i-r]\cup \mathrm{supp}\,(\Phi(m))}$ *is a standard pair of* $I_{\Delta(\Gamma)}$.

Theorem 4.13 is now a corollary of the results stated thus far.

Theorem 4.13. *The iterated Betti numbers of a homogeneous ideal* I *are related to the ideal* $\mathrm{Gin}\,(I)$. *Those of an ideal* I_Γ *are related to the ideals* $\mathrm{Gin}\,(I_\Gamma)$, $I_{\Delta(\Gamma)}$, *and the shifted complex* $\Delta(\Gamma)$. *The relationships are as follows.*

(1) *The multiplicity of* $P_{[i]}$ *with respect to* $\mathrm{Gin}\,(I)$ *is*

$$\mathrm{mult}_{\,\mathrm{Gin}\,(I)}(P_{[i]}) = \sum_r b_{i,r}(I).$$

If $I = I_\Gamma$ *then*

$$\mathrm{mult}_{\,\mathrm{Gin}\,(I_\Gamma)}(P_{[i]}) = \sum_r b_{i,r}(\Gamma) = |\{F \in \max(\Delta(\Gamma)) \,:\, |F| = i\}|.$$

(2) *The degree, geometric degree, and arithmetic degree of* $\mathrm{Gin}\,(I_\Gamma)$ *and* $I_{\Delta(\Gamma)}$ *have the following interpretations:*

$$(i) \qquad \deg(\mathrm{Gin}\,(I_\Gamma)) = \mathrm{geomdeg}\,(\mathrm{Gin}\,(I_\Gamma)) = \sum_r b_{d,r}(I_\Gamma)$$

$$(i') \qquad = \deg(I_{\Delta(\Gamma)}) = |\{F \in \max(\Delta(\Gamma)) \,:\, |F| = d\}| \;;$$

$$(ii) \qquad \mathrm{arithdeg}\,(\mathrm{Gin}\,(I_\Gamma)) = \sum_{i,r} b_{i,r}(I_\Gamma)$$

$$(ii') \qquad = \mathrm{arithdeg}\,(I_{\Delta(\Gamma)}) = |\max(\Delta(\Gamma))|.$$

Equations (i) and (ii) also hold for arbitrary homogeneous ideals I *in* S.

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