Partition Analysis
Geometry, Algebra, Algorithms

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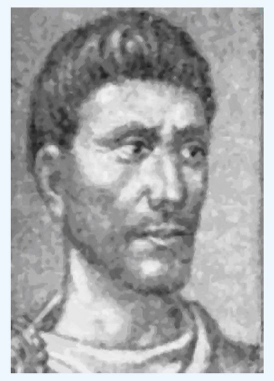
Linear Diophantine Systems and Partition Analysis

A Linear Diophantine System is a linear system $Ax \geq b$ such that $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$ and $x \in \mathbb{N}^n$.

Two problems
- **Counting** (determine the number of $x \in \mathbb{N}^n$ satisfying the system)
- **Parametrized Counting** (determine all $x \in \mathbb{N}^n$ satisfying the system)



Diophantos

- **Solve over** $\mathbb{N}$.
- Αριθμητικά (Arithemtica)
- Fermat's Last Theorem

Example
The 3×3 **symmetric magic squares** with row and column sum equal to r.
Linear Diophantine system:
$a_1,\ldots,a_6,r \geq 0,\ a_1+a_2+a_3-r=0,$
$a_2+a_4+a_5-r=0,\ a_3+a_5+a_6-r=0$

a_1	a_2	a_3
a_2	a_4	a_5
a_3	a_5	a_6

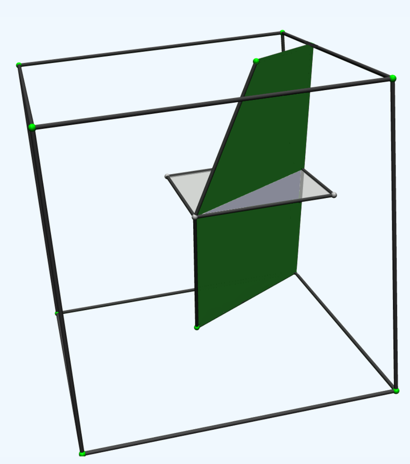
Partition Analysis was introduced by MacMahon in 1915, who applied it to the solution of combinatorial problems subject to linear Diophantine systems using Elliott's algorithm. Towards the end of the last century the method was revived and with the turn of the century Andrews, Paule and Riese gave a completely algorithmic version of Partition Analysis powered by Symbolic Computation.

P.MacMahon

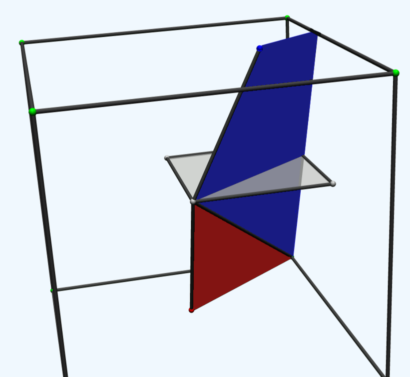
- $\Omega_\geq$ **operator**: $\Omega_\geq \sum_{s_1=-\infty}^{\infty}\cdots\sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r}\lambda_1^{s_1}\cdots\lambda_r^{s_r} := \sum_{s_1=0}^{\infty}\cdots\sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r}$
- Introduce extra variables λ_i that encode the inequalities.
- **Crude Generating Function** for the 3×3 symmetric magic squares:
$$\Omega_= \sum_{a_1,\ldots,a_6,r\geq 0} \lambda_1^{a_1+a_2+a_3-r}\lambda_2^{a_2+a_4+a_5-r}\lambda_3^{a_3+a_5+a_6-r} z_1^{a_1}z_2^{a_2}z_3^{a_3}z_4^{a_4}z_5^{a_5}z_6^{a_6}q^r$$

The Geometry of Elliott's Algorithm

Based on $\dfrac{1}{(1-z_1\lambda^\alpha)(1-\frac{z_2}{\lambda^\beta})} = \dfrac{1}{1-z_1z_2\lambda^{\alpha-\beta}}\left(\dfrac{1}{1-z_1\lambda^\alpha}+\dfrac{1}{1-\frac{z_2}{\lambda^\beta}}-1\right)$, Elliot's Algorithm computes a Partial Fraction Decomposition $\sum_i \dfrac{\pm 1}{\prod_j (1-p_{ij})}$ of the Crude Generating Function for the Homogeneous Problem such that for each i and for all j the exponents of λ in p_{ij} have the same sign. From a geometric viewpoint:



$C = \mathrm{co}((1,0,\alpha),(1,0,-\beta))$
$\rho_C(z) = \dfrac{1}{(1-z_1\lambda^\alpha)(1-\frac{z_2}{\lambda^\beta})}$



$A = \mathrm{co}((1,0,\alpha),(1,1,\alpha-\beta))$ and
$B = \mathrm{co}((1,0,-\beta),(1,1,\alpha-\beta))$.
In terms of generating functions,
$C = A + B - (A \cap B)$ is Elliott's partial fraction decomposition.

After enough iterations we obtain a decomposition into cones that lie either in the non-negative or in the non-positive λ halfspace. We discard the cones that contain generators in the negative λ halfspace and project with respect to the λ-coordinate. The generating functions of the remaining cones sum up to the solution of $\alpha x - \beta y \geq 0$. Generalizing this observation we devise **a geometric algorithm for the evaluation of** $\Omega_\geq$.

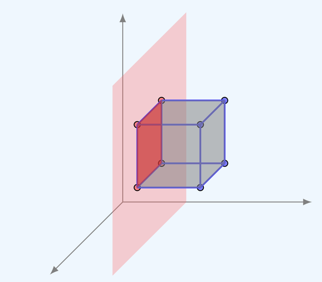
Polyhedra, Cones and their Generating Functions

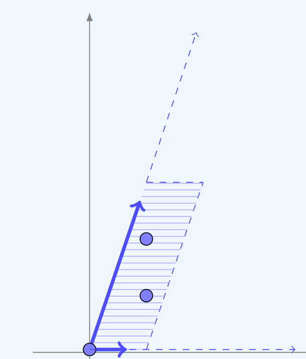
Generating Function

Given a set $S \subseteq \mathbb{N}^n$ we define $\Phi_S(z)$ to be the formal sum $\sum_{s\in S} z^s$, where $z^s = z_1^{s_1}z_2^{s_2}\cdots z_n^{s_n}$. If $\Phi_S(z)$ is a series expansion of a rational function f, then we denote $\rho_S(z) = f$ (the rational generating function).

Polyhdra and Cones

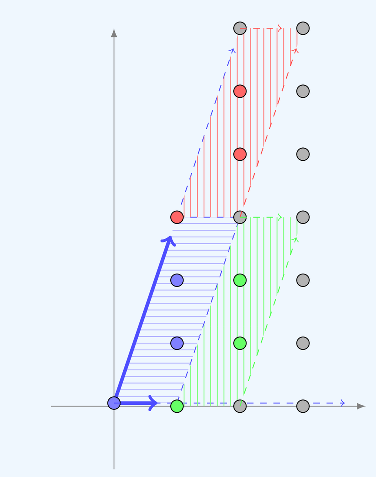
A **polyhedron** is the intersection of finitely many halfspaces in $\mathbb{R}^n$. A bounded polyhedron is called **polytope**. If all the vertices of a polytope are lattice points, then it is called **lattice polytope**.





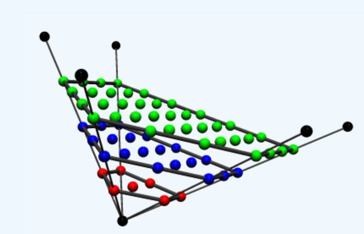
A (polyhedral) **cone** is the intersection of finitely many halfspaces in $\mathbb{R}^n$, whose bounding hyperplanes contain the origin. Equivalently, it is the non-negative span of finitely many vectors in $\mathbb{R}^n$, which are called the **cone generators**. Given a cone $C \in \mathbb{R}^n$ generated by $v_1, v_2, \ldots, v_n$, we define the **fundamental parallelepiped** of C to be $\Pi(C) = \{\sum_{i=1}^n k_i v_i : k_i \in [0,1)\}$. A cone C is called **unimodular** if $\Pi(C) = \{0\}$ and **simplicial** if its generators are linearly independent.

If $C = \mathcal{C}_{\mathbb{R}}(v_1, v_2, \ldots, v_m; 0)$ is simplicial, then $\rho_C(z_1,\ldots,z_n) = \frac{\rho_{\Pi(C)}(z)}{\prod_{i=1}^m(1-z^{v_i})}$. In the picture we see that the fundamental parallelepiped tiles the cone. This translates to multiplying the generating function of the fundamental parallelepiped with the geometric series: $\Phi_S(z_1,z_2) = (1+z_1z_2+z_1z_2^2)\sum_{k,\ell\geq 0} z_1^{k+\ell}z_2^{3\ell}$. Thus, the rational generating function is $\rho_S(z_1,z_2) = \frac{1+z_1z_2+z_1z_2^2}{(1-z_1)(1-z_1z_2^3)}$.



Ehrhart Theory

The t**-dilation** of a set S is defined as $tS = \{tx : x \in S\}$ for $t \in \mathbb{N}$. Let $L_P(t) = |tP \cap \mathbb{Z}^n|$ be a function counting the lattice points in the t-dilate of P. Ehrhart proved that if P is a lattice polytope, then $L_P(t)$ is a degree d polynomial in t, which is called the **Ehrhart polynomial** of P.



Dilations of a pentagon.

One can view the problem of computing the Ehrhart polynomial and the Ehrhart series as a linear Diophantine problem and apply the related algorithmic methods instead of the classic geometric approach of Barvinok.

Graded Rings and Hilbert Series

Let $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{N}^n$. Define the monoid homomorphism $f : \mathbb{N}^n \to \mathbb{Z}^m$ such that $x \mapsto Ax$. Let $\mathcal{R}$ be $\mathbb{K}[z_1, z_2, \ldots, z_n]$ equipped with the degree function induced by the monomial degree $\deg(z^v) = f(v)$. Define $\mathcal{R}_\alpha$ as the $\mathbb{K}$-vector space $\mathbb{K}\{z^v \in \mathcal{R} : \deg(z^v) = \alpha\}$ for $\alpha \in \mathbb{Z}^m$ and B_α to be a basis of $\mathcal{R}_\alpha$ consisting of monic monomials. This defines a grading $\mathcal{R} = \oplus_{\alpha\in\mathbb{Z}^m}\mathcal{R}_\alpha$. Define $F_\alpha(z) = \sum_{p\in B_\alpha} p \in \mathbb{K}[\![z_1, z_2, \ldots, z_n]\!]$ and $H_b(z;\delta) = \sum_{\alpha\geq b} F_\alpha(z)\delta^\alpha \in \mathbb{K}[\![z_1, z_2, \ldots, z_n]\!](\!(\delta_1, \delta_2, \ldots, \delta_m)\!)$. Then:
- $\Phi_{Ax\geq b}(z) = H_b(z;1)$ (substitution meant as taking coefficients)
- $\mathcal{H}(\mathcal{R};\delta) = H_0(1;\delta)$, if all B_α involved are finite and $\mathcal{H}(\mathcal{R};\delta)$ is the classical multigraded Hilbert series.