Reidemeister-Schreier Procedure using Gröbner Bases Techniques

Yvonne Karoske
Fakultät für Informatik und Mathematik, Universität Passau, Germany

UNIVERSITÄT PASSAU
Fakultät für Informatik und Mathematik

Basics

Let $\mathbf{F}(\mathbf{X})$ denote the **free group** on $X = \{x_1, \ldots, x_n\}$.
A word $\omega = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} \in F(X)$ is called **reduced**, if $x_i \neq x_{i+1}$ or $x_i = x_{i+1}$ and $\varepsilon_i \neq -\varepsilon_{i+1}$.
Let G be a group with a generating set X. Using the universal property of free groups one obtains a homomorphism $\varphi : F(X) \to G$, s.t. $\varphi(x) = x \, \forall x \in X$. By the first isomorphism theorem

$$G \simeq F(X) / \ker\varphi.$$

A word $r \in \ker\varphi$ is called a **relator** of G. If $\{r_1, \ldots, r_n\} = R \subseteq \ker\varphi$ with $\ker\varphi = \ll R \gg$ then R is called a set of **defining relators** of G. The set X is called a **generating set** for G. The pair $\langle X \mid R\rangle$ is called a **presentation** of the group G, so

$$G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n\rangle.$$

Schreier Transversal

Let G be an arbitrary finitely presented group and H any subgroup of G. A complete set T of right coset representatives $T = \{g_1, \ldots, g_k\}$ for H in G is called a (right) **Schreier Transversal**, if with every word g_i in T its initial segment is also in T, i.e.

- $g_i = x_1^{\varepsilon_1} \cdots x_i^{\varepsilon_i} \in T \Rightarrow 1, x_1^{\varepsilon_1}, \ldots, x_{i-1}^{\varepsilon_{i-1}} \in T, \quad i = 1, ..., k$
- $Hg_i \neq Hg_j, \quad \forall i \neq j$.

Computing Schreier Transversal

One can find a Schreier Transversal for H in G by **Coset Enumeration** due to the **Todd-Coxeter Procedure** (TC). The procedure enumerates all the cosets systematically and terminates (only in case $[G : H] < \infty$) with a multiplication table of the cosets.

Example: $G = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1\rangle$ and $H < G$, where $H = \langle [a, b]\rangle$.
Then the TC procedure terminates with the Schreier Transversal $T = \{1, a\}$ and the multiplication table:

	a	a^{-1}	b	b^{-1}
1	a	a	a	a
a	1	1	1	1

Schreier's Lemma

Let $G = \langle X\rangle$ be a finitely generated group and suppose that H is a subgroup of G. Let T be a right Schreier Transversal for H in G.
For $g \in G$ let $\overline{g}$ denote the representative in T, i.e. $g \in H\overline{g}$ and set $\gamma(t, x) := tx\overline{tx}^{-1}$, $t \in T$, $x \in X$. Then

$$H = \langle \gamma(t, x) \mid \gamma(t, x) \neq 1\rangle.$$

Reidemeister Rewriting Process

Let $\omega = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$ be a word in H. Define a mapping τ in H, s.t. $\tau(\omega) = S_{t_1,x_1}^{\varepsilon_1} \cdots S_{t_n,x_n}^{\varepsilon_n}$, where

- $\varepsilon_i = 1$: t_i is the coset representative of the initial segment of ω preceding x_i.
- $\varepsilon_i = -1$: t_i is the coset representative of the initial segment of ω upto and including x_i^{-1}

Example:
$\omega = x_1 x_2^{-1} x_3$ then $\tau(\omega) = S_{\overline{1},x_1} \cdot S^{-1}_{\overline{1x_1x_2^{-1}},x_2} \cdot S_{\overline{1x_1x_2^{-1}},x_3}$
The mapping τ is called the **Reidemeister Rewriting Process**.

Reidemeister-Schreier Procedure / Presenting subgroups

Let $G = \langle X \mid R\rangle$ be a finitely presented group and let $H = \langle S\rangle$ be a subgroup of G, where $S \subseteq X$.
Let T be the right Schreier Transversal for H in G. Then one obtains the presentation of H by

$$H = \langle \gamma(t, x) \mid \tau(trt^{-1}), \forall t \in T, \forall r \in R\rangle.$$

Example:
Let $G = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1\rangle$ and $H = \langle [a, b]\rangle$. Then $[G : H] = 2$ and $T = \{1, a\}$. So the presentation is given by

$$\langle x_1, x_2, x_3 \mid x_1 = x_2x_3 = (x_3)^3 = x_3x_2 = (x_2x_1)^2 x_2\rangle.$$

By **Tietze-Transformation** one can simplify the presentation by removing a relation if it can be derived by others and/or removing generators if it is a word of the others.
So H can by simplified to $H = \langle x_1 \mid x_1^3\rangle \simeq \mathbb{Z}/3\mathbb{Z}$.

Research

- In [1] the authors presented a coset enumeration procedure based on prefix Gröbner bases in free group rings and transformed this procedure into a Knuth-Bendix type completion procedure directly comparable to TC.
- Let M be a monoid, K be a field, X be a finite alphabet. In [3] was given a procedure, using the FGLM algorithm, that finds a reduced Gröbner basis (or the right border basis) for the the two sided ideal of the free K-algebra $K\langle X\rangle$ that determines the monoid ring $K[M]$ and therefore gives a presentation for M.

Recent aim of research: giving a full description for the Reidemeister-Schreier procedure using Gröbner bases techniques.

Bibliography

[1] B. Reinert, K. Madlener, T. Mora, **A Note on Nielsen Reduction and Coset Enumeration**, Reports On Computer Algebra No. 19 (1998)

[2] W. Magnus, A. Karras, D. Solitar, **Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations**, John Wiley1966.

[3] M. A. Borges, M. Borges, T. Mora, **Computing Gröbner Bases by FGLM Techniques in a Non-commutative Setting**, J. Symbolic Computation (2000) 30, 429–449