Kähler Differential algebras for finite point sets in $\mathbb{P}^n$

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General setting

Throughout this poster, we let K be an algebraically closed field. Let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s distinct K-rational points in $\mathbb{P}^n$ such that $\mathbb{X} \cap \mathcal{Z}^+(X_0) = \emptyset$. Let $m_1, \dots, m_s$ be positive integers and let $\mathbb{Y}$ be a scheme defined by $I_\mathbb{Y} = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$, where each $\wp_i$ is the defining ideal of the point P_i. The scheme $\mathbb{Y}$, which we denote $\mathbb{Y} = m_1P_1 + \cdots + m_sP_s$, is usually called a set of fat points in $\mathbb{P}^n$. We equip $P = K[X_0, \dots, X_n]$ with its standard grading $\deg(X_i) = 1$. The quotient ring $R_\mathbb{Y} = P/I_\mathbb{Y}$ is the homogenous coordinate ring of the scheme $\mathbb{Y}$.

Definition and Properties

Let $R = \oplus_{i\in\mathbb{Z}} R_i$ be an arbitrary graded K-algebra. Let $\mathcal{J}$ denote the kernel of the canonical multiplication map $\mu : R \otimes_K R \to R, r \otimes r' \mapsto rr'$. The universal derivation of the K-algebra R is the homogeneous K-linear map $d_{R/K} : R \to \mathcal{J}/\mathcal{J}^2$ given by $d_{R/K}(r) = r \otimes 1 - 1 \otimes r + \mathcal{J}^2$, and the module of Kähler differentials of the K-algebra R is the graded R-module $\Omega^1_{R/K} = \mathcal{J}/\mathcal{J}^2$.

Theorem 1. *Let $\mathbb{Y}_1 = m_1P_1 + \cdots + m_sP_s$ and $\mathbb{Y}_2 = (m_1+1)P_1 + \cdots + (m_s+1)P_s$ be two sets of fat points with the corresponding homogeneous ideals $I_{\mathbb{Y}_1}, I_{\mathbb{Y}_2}$ respectively. The sequence of graded $R_{\mathbb{Y}_1}$-modules*

$$0 \longrightarrow I_{\mathbb{Y}_1}/I_{\mathbb{Y}_2} \xrightarrow{\alpha} \Omega^1_{P/K}/I_{\mathbb{Y}_1}\Omega^1_{P/K} \xrightarrow{\varepsilon} \Omega^1_{R_{\mathbb{Y}_1}/K} \longrightarrow 0$$

is exact, where $\alpha(F + I_{\mathbb{Y}_2}) = d_{P/K}F + I_{\mathbb{Y}_1}\Omega^1_{P/K}$ and $\varepsilon(F d_{P/K}H + I_{\mathbb{Y}_1}\Omega^1_{P/K}) = (F + I_{\mathbb{Y}_1}) d_{R_{\mathbb{Y}_1}/K}(H + I_{\mathbb{Y}_1})$.

The $R_{\mathbb{Y}_1}$-module $\Omega^1_{P/K}/I_{\mathbb{Y}_1}\Omega^1_{P/K}$ is a free $R_{\mathbb{Y}_1}$-module of rank $n+1$. By Theorem 1, we get some relations.

- $\mathrm{HF}_{\Omega^1_{R_\mathbb{X}/K}}(i) = (n+1)\,\mathrm{HF}_{\mathbb{Y}_1}(i-1) + \mathrm{HF}_{\mathbb{Y}_1}(i) - \mathrm{HF}_{\mathbb{Y}_2}(i)$.
- The Hilbert polynomial of $\Omega^1_{R_{\mathbb{Y}_1}/K}$ is

$$\mathrm{HP}_{\Omega^1_{R_{\mathbb{Y}_1}/K}} = \sum_{i=1}^{s} (n+2)\binom{m_i+n-1}{n} - \sum_{i=1}^{s}\binom{m_i+n}{n}$$

- If $\mathbb{X} = \{P_1, .., P_s\}$ is in general position in $\mathbb{P}^n$, i.e. no $h+2$ points of $\mathbb{X}$ are on the h-plane for $h < n$. Then the regularity index $r_{\Omega^1_{R_\mathbb{Y}/K}}$ of $\Omega^1_{R_\mathbb{Y}/K}$, is bounded by

$$r_{\Omega^1_{R_\mathbb{Y}/K}} \leq \max\left\{ m_1 + m_2 + 1, [(\sum_{i=1}^{s} m_i + s + n - 2)/n] \right\}.$$

- If $m_1 = \dots = m_s = m \geq 2$ and the set of s distinct K-rational points $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ is a complete intersection, then E. Guardo et al. show that the schemes $Y_i = mP_1 + \cdots + (m-1)P_i + \cdots + mP_s$ all have the same Hilbert functions. Altogether, this show the Hilbert functions of $\Omega^1_{R_{\mathbb{Y}_i}/K}$ are also the same.

Definition 2.

- The exterior algebra over R of the R-module of Kähler differential $\Omega^1_{R/K}$, denoted by $\Omega_{R/K}$, is called Kähler differential algebra of the K-algebra R.
- Let $\Omega^m_{R/K}$ denote the exterior m product over R of R-module $\Omega^1_{R/K}$. Then we have $\Omega_{R/K} = \oplus_{m\in\mathbb{N}}\Omega^m_{R/K}$. The Kähler differential algebra $\Omega_{R/K}$ is said to be bigraded if there exist K-submodules $\Omega^{m,i}_{R/K} \subseteq \Omega^m_{R/K} (m \in \mathbb{N}, i \in \mathbb{Z})$ such that
  - ◇ For each $m \in \mathbb{N}$, $\Omega^m_{R/K} = \oplus_{i\in\mathbb{Z}}\Omega^{m,i}_{R/K}$.
  - ◇ For each $m, m' \in \mathbb{N}$ and $i, i' \in \mathbb{Z}$ then $\Omega^{m,i}_{R/K} \cdot \Omega^{m',i'}_{R/K} \subseteq \Omega^{m+m',i+i'}_{R/K}$.
  - ◇ For each $i \in \mathbb{Z}$, $\Omega^{0,i}_{R/K} = R_i$
  - ◇ For each $r \in R_i, i \in \mathbb{Z}$ then $d_{R/K}(r) \in \Omega^{1,i}_{R/K}$

Proposition 3. *Let $\mathbb{Y} = m_1P_1 + \cdots + m_sP_s$ be a set of fat points in $\mathbb{P}^n$.*

- *If $m_1 = \dots = m_s = 1$ then for each $m > 1$ we have $\dim_K(\Omega^{m,i}_{R_\mathbb{Y}}) = 0$ for all $i \geq 2\sigma_\mathbb{Y} + m + 2$, where $\sigma_\mathbb{Y} = \max\{i \in \mathbb{Z} \mid \mathrm{HF}_\mathbb{Y}(i) < s\}$*
- *If there is an index $j \in \{1, \dots, s\}$ such that $m_j > 1$ then for each $m \leq n+1$ we have*

$$0 < \dim_K(\Omega^{m,i}_{R_\mathbb{Y}/K}) \leq \sum_{i=1}^{s}\binom{n+1}{m}\binom{m_i+n-1}{n}$$

for all $i \geq m$.

Now we let $\mathcal{G}$ be the graded $R_\mathbb{Y}$-module generated by vectors $(\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n})$, where $F \in I_\mathbb{Y}$ and x_i is the image of X_i in $R_\mathbb{Y}$. Set $\deg dx_i = 1$. The exact sequence of graded $R_\mathbb{Y}$-modules

$$0 \longrightarrow \mathcal{G}(-1) \longrightarrow R_\mathbb{Y}^{n+1}(-1) \longrightarrow \Omega^1_{R_\mathbb{Y}/K} \longrightarrow 0,$$

which is mentioned in a paper "Kähler differentials for points in $\mathbb{P}^n$" of G. Dominicis and M. Kreuzer, induces the sequence of K-algebras

$$0 \longrightarrow \mathcal{G} \wedge \bigwedge R_\mathbb{Y}^{n+1} \longrightarrow \bigwedge R_\mathbb{Y}^{n+1} \longrightarrow \Omega_{R_\mathbb{Y}/K} \longrightarrow 0$$

as well as the sequence of graded $R_\mathbb{Y}$-modules

$$0 \longrightarrow \mathcal{G} \wedge \bigwedge^{m} R_\mathbb{Y}^{n+1}(-m-1) \longrightarrow \bigwedge^{m+1} R_\mathbb{Y}^{n+1}(-m-1) \longrightarrow \Omega^{m+1}_{R_\mathbb{Y}/K} \longrightarrow 0 \qquad (1)$$

exact, for each $m \geq 0$. Applying the exact sequence (1), we get the following representation of $\Omega^m_{R_\mathbb{Y}/K}$.

Proposition 4 (Ernst Kunz). *Let $\mathbb{Y}$ be a set of fat points in $\mathbb{P}^n$. Then $\Omega_{R_\mathbb{Y}/K} = \Omega_{P/K}/(I_\mathbb{Y}, dI_\mathbb{Y})$. In particularly, for each $m \in \mathbb{N}$ we have*

$$\Omega^m_{R_\mathbb{Y}/K} = \Omega^m_{P/K}/(I_\mathbb{Y}\Omega^m_{P/K} + dI_\mathbb{Y}\Omega^{m-1}_{P/K}).$$

By using above presentation, I can write an ApCoCoA function which take the homogeneous ideal of a set of fat points $\mathbb{Y}$ and a number m in $\mathbb{N}$ as input and compute the values of the Hilbert function of $\Omega^m_{R_\mathbb{Y}/K}$. Moreover, the Hilbert function of $\Omega^m_{R_\mathbb{Y}/K}$ is described by the next proposition.

Proposition 5. *Let $\alpha_\mathbb{Y} = \min\{i \in \mathbb{Z} \mid (I_\mathbb{Y})_i \neq 0\}$, and $\rho_{\mathbb{Y},m}$ be the regularity index of $\Omega^m_{R_\mathbb{Y}/K}$.*

- *For $i < m$ we have $\dim_K(\Omega^{m,i}_{R_\mathbb{Y}/K}) = 0$.*
- *For $m \leq i < \alpha_\mathbb{Y} + m - 1$ then $\mathrm{HF}_{\Omega^m_{R_\mathbb{Y}/K}}(i) = \binom{n+1}{m} \cdot \binom{n+i-m}{n}$.*
- *We have $\mathrm{HF}_{\Omega^m_{R_\mathbb{Y}/K}}(\sigma_\mathbb{Y} + m + 1) \geq \mathrm{HF}_{\Omega^m_{R/K}}(\sigma_\mathbb{Y} + m + 2) \geq \cdots$ and if $\rho_{\mathbb{Y},m} \geq \sigma_\mathbb{X} + m + 1$ then $\mathrm{HF}_{\Omega^m_{R_\mathbb{Y}/K}}(\sigma_\mathbb{Y} + m + 1) > \mathrm{HF}_{\Omega^m_{R_\mathbb{Y}/K}}(\sigma_\mathbb{Y} + m + 2) > \cdots > \mathrm{HF}_{\Omega^m_{R_\mathbb{Y}/K}}(\rho_{\mathbb{Y},m})$.*

The following example shows that $\mathrm{HF}_{\Omega^m_{R/K}}(i)$ may or may not be monotonic in the range of $\alpha_\mathbb{X} + m \leq i \leq \sigma_\mathbb{X} + m + 1$. Let $\mathbb{X} \subset \mathbb{P}^2$ consist of following nine points $\{(1:0:0), (1:0:1), (1:0:2), (1:0:3), (1:0:4), (1:0:5), (1:1:0), (1:2:0), (1:1:1)\}$. Then

- $\mathrm{HF}_\mathbb{X}$: 1 3 6 7 8 9 9, $\cdots$ $\alpha_\mathbb{X} = 3$ and $\sigma_\mathbb{X} = 4$.
- $\mathrm{HF}_{\Omega^1_{R/K}}$: 0 3 9 15 14 13 14 13 12 11 10 9 9 $\cdots$
- $\mathrm{HF}_{\Omega^2_{R/K}}$: 0 0 3 9 9 4 5 4 3 2 1 0 0 $\cdots$
- $\mathrm{HF}_{\Omega^3_{R/K}}$: 0 0 0 1 3 0 0 $\cdots$

Some special cases

Proposition 6. *Let $\mathbb{X} = \{P_1 = (P_{10} : P_{11} : \cdots : P_{1n}), \dots, P_s = (P_{s0} : P_{s1} : \cdots : P_{sn})\}$ be a set of s distinct K-rational points in $\mathbb{P}^n$. Let $\mathcal{A} = (P_{ij}) \in \mathrm{Mat}_{s,n+1}(K)$ and let r be the rank of the matrix $\mathcal{A}$. Then we have $\Omega^{r+i}_{R_\mathbb{X}/K} = 0$ for all $i \geq 1$ and $\mathrm{HF}_{\Omega^r_{R_\mathbb{X}/K}}(r) = 1$.*

For example, let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s distinct K-rational points on a line in $\mathbb{P}^n$. Then $\Omega^3_{R_\mathbb{X}/K} = 0$. Moreover, we also have

- $\mathrm{HF}_{\Omega^1_{R_\mathbb{X}/K}}$: 0 2 4 6 ... $2(s-2)$ $2(s-1)$ $2s-1$ $2s-2$ ...s s ...
- $\mathrm{HF}_{\Omega^2_{R_\mathbb{X}/K}}$: 0 0 1 2 ... $s-2$ $s-1$ $s-2$ $s-3$ ... 0 0 ...

We can characterize the figuration of a set of s distinct K-rational points $\mathbb{X}$ in $\mathbb{P}^2$ by looking at the values of the Hilbert function of $\Omega^3_{R_\mathbb{X}/K}$ as follows.

Proposition 7. *Let $\mathbb{X}$ be a set of s distinct K-rational points in $\mathbb{P}^2$ with $s > 4$. Then we have*

- *$\mathrm{HF}_{\Omega^3_{R_\mathbb{X}/K}}(i) = 0$ for all $i \in \mathbb{N}$ if and only if $\mathbb{X}$ lies on a line.*
- *$\mathrm{HF}_{\Omega^3_{R_\mathbb{X}/K}}(i) \leq 1$ for all $i \in \mathbb{N}$ and $\mathrm{HF}_{\Omega^3_{R_\mathbb{X}/K}}(3) = 1$ if and only if $\mathbb{X}$ lies on a quadric. Further more,*
  - ◇ *If $\mathrm{HF}_{\Omega^3_{R_\mathbb{X}/K}}(3) = 1, \mathrm{HF}_{\Omega^3_{R_\mathbb{X}/K}}(4) = 1$, then $\mathbb{X}$ lies on two lines, none of $s-1$ points of $\mathbb{X}$ lies on a line.*
  - ◇ *If $\mathrm{HF}_{\Omega^3_{R_\mathbb{X}/K}}(3) = 1, \mathrm{HF}_{\Omega^3_{R_\mathbb{X}/K}}(i) = 0, i \neq 3$ and $\Delta\mathrm{HF}_\mathbb{X}(i) \leq 1$ for all $i \geq 2$, then $\mathbb{X}$ contains $s-1$ points which lie on a line.*
  - ◇ *If $\mathrm{HF}_{\Omega^3_{R_\mathbb{X}/K}}(3) = 1, \mathrm{HF}_{\Omega^3_{R_\mathbb{X}/K}}(i) = 0, i \neq 3$ and there is $i \geq 2$ such that $\Delta\mathrm{HF}_\mathbb{X}(i) \geq 2$ then $\mathbb{X}$ lies on an irreducible quadric.*

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