On the number of generators of a projective module

Shiv Datt Kumar
(Joint Work with Sumit Kumar Upadhyay and Raja Sridharan)
Department of Mathematics
Motilal Nehru National Institute of Technology, Allahabad, India

Abstract

In this article we give a bound on the number of generators of a finitely generated projective module of constant rank over a commutative Noetherian ring in terms of the rank of module and the dimension of ring. Under certain conditions we provide an improvement to the Forster - Swan bound in case of finitely generated projective modules of rank n over an affine algebra over a finite field or an algebraically closed field.

Notations

R = commutative Noetherian ring with identity,
$M_{m\times n}(R)$ = the set of all $m \times n$ matrices over ring R.
M = finitely generated R- modules and P & Q = finitely generated projective R-modules.
$\mu(M)$ = the minimum number of elements needed to generate M over R

Introduction

One can ask: Is $\mu(M)$ bounded? Forster [5] and Swan [7] have given an upper bound on the number of generators of M in terms of local information i.e. $\mu(M) = sup\{\mu(Mp) + dim(R/p) \mid p \in spec(R)\}$. Eisenbud and Evans [2] established the stability of this bound in the sense that, given any finite set of generators, a set of predicted size can be obtained by certain elementary operations. There are certain class of rings R over projective R module P is free. But if P is not free, we have a natural question what is the bound on the number of generators? The main aim of this paper is to prove a result on elementary matrices which connects two sets of completable unimodular rows. By using this result we deduce a bound on the number of generators of P in terms of rank of P and dimension of R. Under certain conditions show that our methods actually provide an improvement to the Forster - Swan bound if P is a module over an affine algebra over a finite field or an algebraically closed field. Its proof has merit that it can be made algorithmic.

Preliminaries

Theorem 1.1. [9] Let $A \in M_{m\times n}(R)$ and $\mu_A : R^n \longrightarrow R^m$ be an R-module homomorphism given by $\mu_A(X) = AX$, where $n \geq m$. Then μ_A is surjective if and only if $I_m(A) = R$, where $I_m(A)$ denotes the ideal generated by all $m \times m$ minors of A.

Lemma 1.2. Let $A = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ and $\mu_A : R^n \longrightarrow R^m$ be the R-module homomorphism given by $\mu_A(X) = AX$, where $n \geq m$. Then μ_A is surjective if and only if the columns of A generate R^m.

Proposition 1.3. [3] Let R be a semilocal ring and P a finitely generated R-module of constant rank. Then P is free.

Proposition 1.4. [6] Let R be of finite dimension and s is a non-zero divisor in R. Then $dim(R_{s\langle 1+sR\rangle}) < dim(R)$.

Proposition 1.5. Let S be the set of all non-zero divisors of R. If $S^{-1}P = \langle \frac{c_1}{s_1}, \cdots, \frac{c_k}{s_k}\rangle$, $s_i \in S (1 \leq i \leq k)$. Then there exists $s \in S$ such that $P_s = \langle \frac{c_1}{s_1}, \cdots, \frac{c_k}{s_k}\rangle$.

Theorem 1.6. [8] Suppose $dim(R) = d$. Then for $n \geq d+2$, any unimodular row of length n over R is completable.

Lemma 1.7. [6] Let $s \in R$ be a non zero divisor in R. Suppose there exist two surjections $f : R_s^n \longrightarrow P_s$ and $g : R_{1+sR}^n \longrightarrow P_{1+sR}$. If there exists $\sigma \in GL_n(R_{s(1+sR)})$ such that $matrix(\hat{f})\sigma = matrix(\hat{g})$ (where $\hat{f} : R_{s(1+sR)}^n \longrightarrow P_{s(1+sR)}$ and $\hat{g} : R_{s(1+sR)}^n \longrightarrow P_{s(1+sR)}$ are induced by f and g respectively) and further that $\sigma = \tau_1\tau_2$, where $\tau_1 \in GL_n(R_s)$ and $\tau_2 \in GL_n(R_{1+sR})$, then P is generated by n elements.

Lemma 1.8. [6] Let $s, t \in R$ be non zero divisors such that $sR + tR = R$ and $\sigma \in GL_n(R_{st})$ be such that σ can be connected to the identity matrix. Then $\sigma = \tau_1\tau_2$ for some $\tau_1 \in GL_n(R_s)$ and $\tau_2 \in GL_n(R_t)$.

Results

Theorem 2.1. Assume $dim(R) = d$. Suppose R-module $\langle \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, \cdots, \begin{pmatrix} a_n \\ b_n \end{pmatrix}\rangle$ generate R^2, where $n \geq d+3$. Then there exists $\sigma \in E_n(R)$ such that

$$\sigma \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}.$$

This theorem follows from Theorem 1.1, Lemma 1.2 and Theorem 1.6. Now we state general form of Theorem 2.1.

Theorem 2.2. Assume $dim(R) = d$.. Suppose $\langle \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}\rangle = R^m$. Then there exists $\sigma \in E_n(R)$ such that

$$\sigma \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$$ if $n \geq d+m+1$, where I_m and 0 denote $m \times m$ identity matrix and $(n-m) \times m$ zero matrix respectively.

In other words if $\phi : R^n \longrightarrow R^m$ is surjective homomorphism, then there exists $\sigma \in E_n(R)$ such that $matrix(\phi)\sigma = \begin{pmatrix} I_m & 0 \end{pmatrix}$ if $n \geq d+m+1$, where I_m and 0 denote $m \times m$ identity matrix and $m \times (n-m)$ zero matrix respectively.

As an application of Theorem 2.2, we prove Theorem 2.3 which also follows from the theorem of Forster and Swan ([5] & [7]).

Theorem 2.3. Let R be a reduced Noetherian ring of dimension d and rank$(P) = n$. Then P is generated by $n+d$ elements over R.

Proof: Take $S = R - \cup_{i=1}^{l} p_i$, where p_i are minimal prime ideal of R. Then $S^{-1}P$ is a free $S^{-1}R$ module of rank n. Hence there exists $s \in S$ such that P_s is free of rank n. We prove the theorem by induction on dimension of R. By induction hypothesis we can assume that P/sP is generated by $n+d$ elements. Therefore we have a surjective homomorphism $f : R_{1+sR}^{n+d} \longrightarrow P_{1+sR}$. Since P_s is generated by n elements, we have a surjective homomorphism $g : R_s^{n+d} \longrightarrow P_s$. Consider the induced surjective homomorphisms $\hat{f} : R_{s(1+sR)}^{n+d} \longrightarrow P_{s(1+sR)}$ and $\hat{g} : R_{s(1+sR)}^{n+d} \longrightarrow P_{s(1+sR)}$ of f and g respectively. Since $dim(R_{s(1+sR)}) < dim(R)$, $n+d > n + dim(R_{s(1+sR)})$. This implies that $n+d \geq dim(R_{s(1+sR)}) + n + 1$. So from Theorem 2.2, there exists $\sigma \in E_{n+d}(R_{s(1+sR)})$ such that $matrix(\hat{f})\sigma = matrix(\hat{g})$. Form Lemma 1.7 and Lemma 1.8, P is generated by $n+d$ elements over R.

Theorem 2.4. Let R be a non-reduced Noetherian ring of dimension d and rank$(P) = n$. Then P is generated by $n+d$ elements over R.

Proof: Let $R' = R/Nil(R)$ and $P' = P/Nil(R)P$. Then $dim(R') = d$ and rank$(P') = n$. By Theorem 2.3, $P/Nil(R)P$ is generated by $n+d$ elements. So by Nakayama lemma, P is generated by $n+d$ elements.

Note. By Theorem 2.4, it is clear that P is generated by $n+d$ elements over any R. Now we prove a theorem which provides improvement in the previous bound under certain conditions.

Theorem 2.5. Assume $dim(R) = d$ and rank$(P) = n \geq 2$. Suppose all finitely generated projective module of rank d have a unimodular element. Then P is generated by $n+d-1$ elements over R.

Proof: From Theorem 2.4, P is generated by $n+d$ elements i.e. $P \oplus Q \cong R^{n+d}$. Hence $rank(Q) = d$. By assumption Q has a unimodular element say q. Therefore q is also a unimodular element in R^{n+d}. Since $n \geq 2$, $n+d \geq d+2$. So by Theorem 1.6, q is elementary completable. Thus $R^{n+d}/\langle q\rangle \cong R^{n+d-1}$. Therefore we have a surjective map from R^{n+d-1} to P. This copmpletes the proof.

Conti...

Corollary 2.6. Let R be an affine algebra of dimension d over an algebraically closed field such that all finitely generated projective R-modules of constant rank d have a unimodular element and rank$(P) = n$. Then P is generated by $n+d-1$ elements over R.

Proof: In [1] Suslin has shown that stably free R - module of rank $\geq dim(R)$ over an affine algebra R over an algebraically closed field is free. Hence any unimodular row of length $\geq d+1$ is completable. Then the proof follows from Theorem 2.5.

Corollary 2.7. Let R be an affine algebra of dimension d over $\bar{F_p}$, where $\bar{F_p}$ denotes the algebraic closure of the field of p elements such that all finitely generated projective R-modules of constant rank d have a unimodular element and rank$(P) = n$. Then P is generated by $n+d-1$ elements over R.

Proof: In [4] Mohan Kumar, Murthy and Roy have shown that stably free R - module of rank $\geq dim(R)$ over an affine algebra R over $\bar{F_p}$ is free. Hence any unimodular row of length $\geq d+1$ is completable. Then the proof follows from Theorem 2.5.

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