Castelnuovo-Mumford regularity and Gorensteinness of fiber cone

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Abstract

We study the Castelnuovo-Mumford regularity and Gorenstein properties of the fiber cone. We obtain upper bounds for the Castelnuovo-Mumford regularity of the fiber cone and obtain sufficient conditions for the regularity of the fiber cone to be equal to that of the Rees algebra. We give a formula for the canonical module of the fiber cone and use it to give a criterion for a Cohen-Macaulay fiber cone to be Gorenstein under the assumption that the associated graded ring is Gorenstein. Finally we discuss many interesting consequences of this criterion.

Definitions

Let $(A,\mathfrak{m})$ be a commutative Noetherian local ring and I be an ideal of A. The graded algebras:

$R(I) := \oplus_{n\geq 0} I^n t^n \subset R[t]$, the **Rees algebra**
$G(I) := \oplus_{n\geq 0} I^n/I^{n+1}$, the **associated graded ring**
$F(I) := \oplus_{n\geq 0} I^n/\mathfrak{m}I^n$, the **fiber cone**.

These algebras are known as the **blowup algebras** associated to I.

For a standard graded algebra $S = \oplus_{n\geq 0} S_n$ over a commutative Noetherian ring S_0 and a finitely generated graded S-module $M = \oplus_{n\geq 0} M_n$, define

$$a(M) := \begin{cases} \max\{n \mid M_n \neq 0\} & \text{if } M \neq 0 \\ -\infty & \text{if } M = 0. \end{cases}$$

For $i \geq 0$, set

$$a_i(M) := a(H^i_{S_+}(M)),$$

where S_+ denotes the ideal of S generated by the homogeneous elements of positive degree and $H^i_{S_+}(M)$ denotes the i-th local cohomology module of M with respect to the ideal S_+.

The ***Castelnuovo-Mumford regularity*** (or *regularity*) of M is defined as the number

$$\operatorname{reg}(M) := \max\{a_i(M) + i \mid i \geq 0\}.$$

A ***canonical module*** of a Cohen-Macaulay graded ring S is a maximal Cohen-Macaulay graded module of finite injective dimension. It is unique up to a graded isomorphism, and denote by ω_S.

A Cohen-Macaulay ring S with canonical module ω_S is said to be ***Gorenstein*** if $\omega_S \cong S(-b)$, for some $b \in \mathbb{Z}$.

The dimension of $F(I)$ is called the ***analytic spread*** of I and denote by ℓ.

$$\mathcal{F} : A \supset \mathfrak{m} \supset \mathfrak{m}I \supset \mathfrak{m}I^2 \supset \cdots$$

and $R(\mathcal{F}) := A \oplus \mathfrak{m}t \oplus \mathfrak{m}It^2 \oplus \mathfrak{m}I^2t^3 \oplus \cdots$.

Castelnuovo-Mumford regularity of fiber cone

Theorem. *Let $(A,\mathfrak{m})$ be a Noetherian local ring and I be an ideal of A with $\ell = 1$. Then*

$$\operatorname{reg} F(I) \leq \operatorname{reg} G(I).$$

Furthermore, if $\operatorname{grade} I = 1$, then $\operatorname{reg} F(I) = \operatorname{reg} G(I) = r(I)$, where $r(I)$ denotes the reduction number of I.

Theorem. *Let $(A,\mathfrak{m})$ be a Noetherian local ring and I be an ideal of A. Suppose $\operatorname{grade} I = \ell$ and $\operatorname{grade} G(I)_+ \geq \ell - 1$. Then*

$$\operatorname{reg} F(I) \geq \operatorname{reg} G(I).$$

Furthermore, if $\operatorname{depth} F(I) \geq \ell - 1$, then $\operatorname{reg} F(I) = \operatorname{reg} G(I)$.

Some sufficient conditions for the equality of $\operatorname{reg} F(I)$ and $\operatorname{reg} G(I)$:

Proposition. *Let $(A,\mathfrak{m})$ be a Noetherian local ring and I be an ideal of A. If $\operatorname{reg} R(\mathcal{F}) \leq \operatorname{reg} R(I)$, then*

$$\operatorname{reg} F(I) = \operatorname{reg} R(I).$$

Proposition. *Let $(A,\mathfrak{m})$ be a Noetherian local ring and I be an $\mathfrak{m}$-primary ideal of A such that $\operatorname{grade} I > 0$. Suppose*

$$I^{n_0} = \mathfrak{m}I^{n_0 - 1}$$

for some $n_0 \in \mathbb{N}$. Then

$$\operatorname{reg} F(I) = \operatorname{reg} G(I).$$

Theorem. *Let $(A,\mathfrak{m})$ be a Noetherian local ring and I be an ideal of A such that $\operatorname{grade} I > 0$. Assume that $\mathfrak{m}G(I)$ is a Cohen-Macaulay $R(I)$-module of dimension ℓ. Then*

- $\operatorname{reg} F(I) \leq \operatorname{reg} R(I)$;
- *if $a_\ell(R(\mathcal{F})) - 1 < a_\ell(R(I))$, then $\operatorname{reg} F(I) = \operatorname{reg} R(I)$;*
- *if $a_\ell(R(\mathcal{F})) - 1 = a_\ell(R(I))$, then $\operatorname{reg} \mathfrak{m}G(I) \leq \operatorname{reg} R(I)$ and $\operatorname{reg} F(I) \leq \operatorname{reg} R(I)$. Furthermore, if $\operatorname{reg} \mathfrak{m}G(I) < \operatorname{reg} G(I)$, then $\operatorname{reg} F(I) = \operatorname{reg} R(I)$.*

Example. *Let $A = k[\![X,Y,Z]\!]/J$, where $J = (X^4, XY^2Z, XYZ^2, YZ^4, Z^5)$. Let $I = (\bar{X}^3, \bar{Y}^2, \bar{Z}^2)$ and $\mathfrak{m} = (\bar{X}, \bar{Y}, \bar{Z})$. Then the reduction number of I is $r(I) = 3$ and $\operatorname{reg}(F(I)) = 4$. Therefore $r(I) < \operatorname{reg} F(I)$.*

Gorenstein fiber cones

Proposition. *Let $(A,\mathfrak{m})$ be a Noetherian local ring and I be an $\mathfrak{m}$-primary ideal such that the associated graded ring $G(I)$ is Cohen-Macaulay. Let $\omega_{G(I)} = \oplus_{n\in\mathbb{Z}} \omega_n$ and $\omega_{F(I)}$ be the canonical modules of $G(I)$ and $F(I)$ respectively. Then*

- *$\omega_{F(I)} \cong \oplus_{n\in\mathbb{Z}} (0 :_{\omega_n} \mathfrak{m})$;*
- *$a(F(I)) = a(G(I)) = r - d$, where r is the reduction number of I with respect to any minimal reduction J of I;*
- *for any $k \in \mathbb{N}^{r+1}$, $a(F(I^k)) = [\frac{a(F(I))}{k}] = [\frac{r-d}{k}]$;*
- *if $G(I)$ is Gorenstein, then*

$$\omega_{F(I)} \cong \bigoplus_{n\in\mathbb{Z}} \frac{(I^{n+r-d+1} : \mathfrak{m}) \cap I^{n+r-d}}{I^{n+r-d+1}}.$$

Corollary. *Suppose $G(I)$ and $F(I)$ are Cohen-Macaulay. Then $\operatorname{reg} \omega_{G(I)} = \operatorname{reg} \omega_{F(I)}$. In addition if $G(I)$ or $F(I)$ is Gorenstein then*

$$\operatorname{reg} \omega_{G(I)} = \dim(A) = \operatorname{reg} \omega_{F(I)}.$$

A Criterion for a Cohen-Macaulay fiber cone to be Gorenstein, if the associated graded ring is Gorenstein:

Theorem. *Let $(A,\mathfrak{m})$ be a Noetherian local ring, I be an $\mathfrak{m}$-primary ideal and J be a minimal reduction of I with reduction number r. Assume that $G(I)$ is a Gorenstein ring and $F(I)$ is Cohen-Macaulay. Then $F(I)$ is Gorenstein if and only if*

$$\lambda\left(\frac{((I^{n+1}+J) : \mathfrak{m}) \cap I^n}{I^{n+1} + JI^{n-1}}\right) = \lambda\left(\frac{I^n}{\mathfrak{m}I^n + JI^{n-1}}\right)$$

for $0 \leq n \leq r$.

Some interesting consequences of the criterion:

Corollary. *Let $(A,\mathfrak{m})$ be a Noetherian local ring and I is an $\mathfrak{m}$-primary ideal. Then $G(I)$ and $F(I)$ are Gorenstein $\Rightarrow A/I$ is Gorenstein.*

Corollary. *Suppose I is an $\mathfrak{m}$-primary ideal such that $G(I)$ and $F(I)$ are Gorenstein. Then $\operatorname{pd}(A/I) < \infty \Leftrightarrow I$ is generated by a regular sequence.*

Corollary. *Suppose $(A,\mathfrak{m})$ is a regular local ring and I is an $\mathfrak{m}$-primary ideal of A such that $G(I)$ and $F(I)$ are Gorenstein. Then I is generated by a regular sequence.*

References

[1] A.V. Jayanthan and Ramakrishna Nanduri, Castelnuovo-Mumford regularity and Gorensteinness of fiber cone. *Communications in Algebra*, **40**(4) 2012, 1338–1351.