Monomial ideals associated to bipartite graphs with Castelnuovo-Mumford regularity 3

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Abstract

Studying homological invariants of monomial ideals in a polinomial ring $R = \mathbb{K}[x_1, \dots, x_n]$ by looking for combinatorial properties in discrete objects associated to them (like graphs, hypergraphs, simplicial complexes,...) is a well known technique that has been fruitfully exploited in the last decades. In fact, it has stablished a quite complete dictionary between these algebraic and combinatoric classes.

In this work we focus on squarefree monomial ideals whose associated combinatorial objects are bipartite graphs. For this class, we characterize those having Castelnuovo-Mumford regularity 3 in terms of induced cycles in the bipartite complement of the associated graph. When the regularity is > 3, we determine the first Betti number on the 4-th row of the Betti diagram.

Definitions

Every homogeneous ideal I in R has a finite minimal graded free resolution

$$0 \longrightarrow \bigoplus_j R(-j)^{\beta_{l,j}} \longrightarrow \dots \longrightarrow \bigoplus_j R(-j)^{\beta_{1,j}} \xrightarrow{\varphi_1} \bigoplus_j R(-j)^{\beta_{0,j}} \xrightarrow{\varphi_0} I \longrightarrow 0$$

and its graded Betti numbers $\beta_{i,j}$ are arranged in the so called **Betti diagram**.

	0	...	i	...	l
2	$\beta_{0,2}$	...	$\beta_{i,2+i}$	...	$\beta_{l,2+l}$
...	...	...	...	...	...
r	$\beta_{0,r}$	...	$\beta_{\mathbf{i,r+i}}$	...	$\beta_{l,r+l}$
...	...	...	...	...	...
$r_{\max}$	$\beta_{0,r_{\max}}$	...	$\beta_{i,r_{\max}+i}$	...	$\beta_{l,r_{\max}+l}$

The **Castelnuovo-Mumford regularity**, that can be defined as

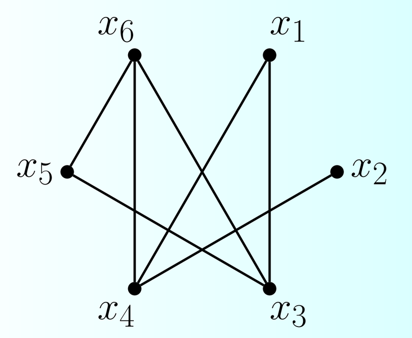
$$\mathbf{reg(I)} = max\{j - i : \beta_{i,j} \neq 0\}$$

coincides with the biggest label of a row containing a non-zero Betti number in the diagram.

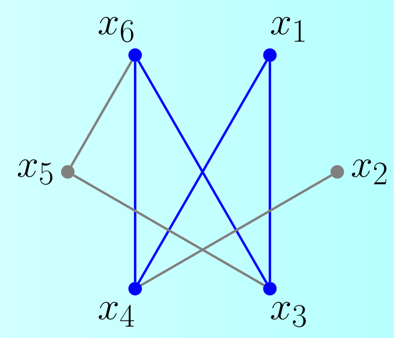
A monomial ideal I in R generated by squarefree monomials of degree 2, i. e., $I = (x_{i_1}x_{j_1}, \dots, x_{i_s}x_{j_s})$ where $i_l \neq j_l$, is called an **edge ideal** as it can be associated to the graph $\mathbf{G_I}$ whose vertex and edge sets are $V(G_I) = \{x_1, \dots, x_n\}$ and $E(G_I) = \{\{x_{i_1}, x_{j_1}\}, \dots, \{x_{i_s}, x_{j_s}\}\}$, respectively.

An **induced cycle** C in a graph G is a subgraph on the vertices $\{x_{k_1}, \dots, x_{k_t}\}$, with $t \geq 4$, such that $E(G) = \{\{x_{k_1}, x_{k_2}\}, \{x_{k_2}, x_{k_3}\}, \dots, \{x_{k_{t-1}}, x_{k_t}\}, \{x_{k_t}, x_{k_1}\}\}$ and $\{x_{k_i}, x_{k_j}\} \notin E(G)$ if $\{x_{k_i}, x_{k_j}\} \notin E(C)$. We say that t is the **length of the cycle**.

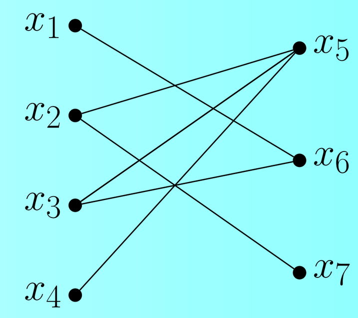
A graph G is said to be **bipartite** if its vertex set can be splitted into two disjoint sets, $V(G) = A \sqcup B$, in such a way that $x_{i_l} \in A$ and $x_{j_l} \in B, \forall l = 1, \dots, s$.



G_I for $I = (x_1x_3, x_1x_4, x_2x_4, x_3x_5, x_3x_6, x_4x_6, x_5x_6)$



Induced cycle in G_I



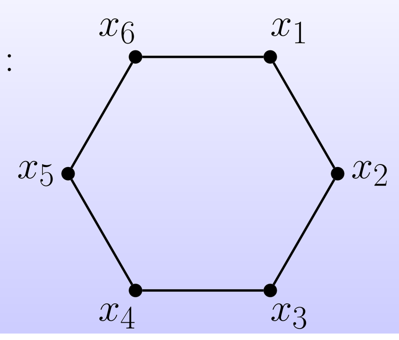
Bipartite graph

Regularity 2 for Edge Ideals (known results)

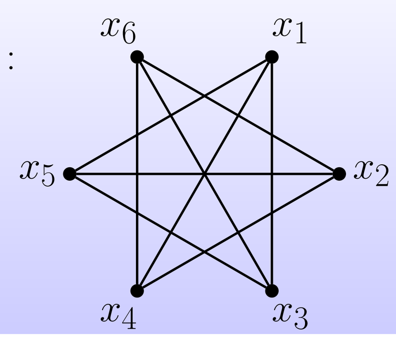
Definition. *The* **complement** *of a given graph G is the graph $\mathbf{G^c}$ on the same vertex set as G with $E(G^c) = \{\{u, v\} : u, v \in V(G), \{u, v\} \notin E(G)\}$.*

Example. *The complement of a cycle C_t of length t is denoted by $\mathbf{C_t^c}$.*

$\mathbf{C_6}$:



$\mathbf{C_6^c}$:



Betti diagram of I s. t. $G_I = C_6^c$

	0	1	2	3
2	9	16	9	-
3	-	-	-	1

Betti numbers of ideals associated to these graphs are completely described below:

Proposition ([2]). *If $G_I = C_t^c$, then $reg(I) = 3$ and the minimal graded free resolution of I is*

$$0 \longrightarrow R(-t) \longrightarrow R(-t+2)^{\beta_{t-4,t-2}} \longrightarrow \dots \longrightarrow R(-2)^{\beta_{0,2}} \longrightarrow I \longrightarrow 0$$

where $\beta_{i,i+2} = t\frac{i+1}{t-i-2}\binom{t-2}{i+2}$ for all $0 \leq i \leq t-4$.

Fröberg gave a characterization of edge ideals having regularity 2:

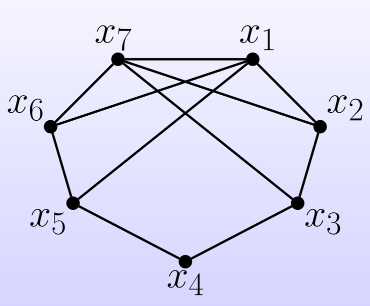
Theorem ([3]). *An edge ideal I has regularity 2 if and only if G_I^c has no induced cycle of length greater than or equal to 4.*

This theorem has been refined computing extra information when $reg(I) > 2$:

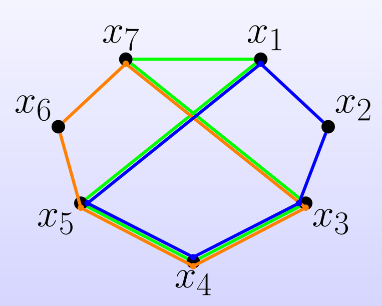
Theorem ([1,2]). *Let I be an edge ideal with $reg(I) > 2$ and r be the minimal length of an induced cycle in G_I^c. Then*

- *$\beta_{i,j} = 0$ if $i < r-3$ and $j > i+2$;*
- *$\beta_{i,j} = 0$ if $i = r-3$ and $j > i+3$;*
- *$\beta_{r-3,r} = \#(\{$induced cycles in $G_I^c$of length $r\})$.*

Example. *Let $I = (x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_2x_6, x_3x_5, x_3x_6, x_4x_6, x_4x_7, x_5x_7)$*



G_I^c



3 induced 5-cycles in G_I^c

	0	1	2	3	4
2	10	19	11	1	-
3	-	-	3	4	1

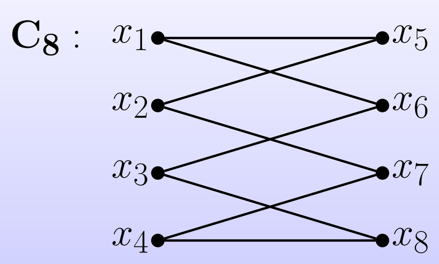
Betti diagram of I

Regularity 3 in the Bipartite Case (new results)

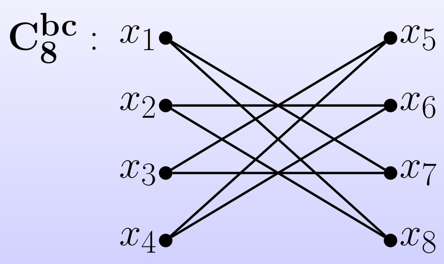
Definition. *The* **bipartite complement** *of a given bipartite graph G with $V(G) = A \sqcup B$ is the (bipartite) graph $\mathbf{G^{bc}}$ on the same vertex set with $E(G^{bc}) = \{\{u, v\} : u \in A, v \in B, \{u, v\} \notin E(G)\}$.*

Example. *The bipartite complement of a cycle of length t, where t must be even, is set $\mathbf{C_t^{bc}}$*

$\mathbf{C_8}$:



$\mathbf{C_8^{bc}}$:



Betti diag. for $G_I = C_8^{bc}$

	0	1	2	3	4
2	8	8	-	-	-
3	-	12	24	12	-
4	-	-	-	-	1

Betti numbers of ideals associated to these graphs are completely described below:

Proposition. *If $G_I = C_t^{bc}$ with $t = 2s$, then $reg(I) = 4$ and*

- $\beta_{j-2,j} = \sum_{k=1}^{j-1}\sum_{c=1}^{k}\frac{s}{c}\binom{k-1}{c-1}\binom{s-k-1}{c-1}\binom{s-k-c}{j-k}$, $j = 2, \dots, s-1$;
- $\beta_{j-3,j} = \sum_{m=2}^{\lfloor j/2 \rfloor}(t\frac{m-1}{m}\binom{2n-j-1}{m-1}\sum_{a=0}^{j-2m}\binom{j-m-a-1}{m-1}\binom{2n-j-m}{a}), j = 4, \dots, t-2$;
- $\beta_{t-4,t} = 1$.

Similarly to Froberg's theorem, we can characterize ideals associated to bipartite graphs with $reg(I) \leq 3$:

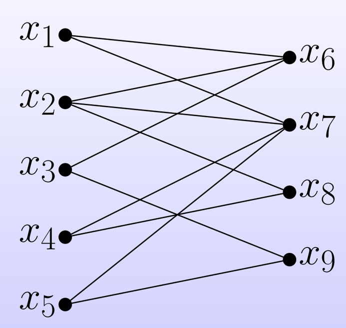
Theorem. *An edge ideal I whose associated graph G_I is bipartite has regularity ≤ 3 if and only if G_I^{bc} has no induced cycle of length greater than or equal to 6.*

When $reg(I) > 3$ we also describe some properties of the Betti diagram:

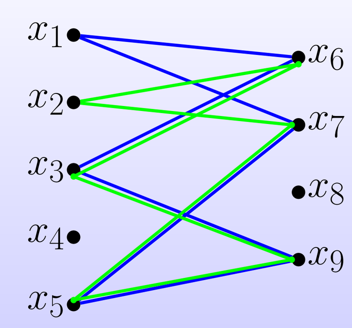
Theorem. *Let I be an edge ideal such that G_I is bipartite and $reg(I) > 3$. Set $r = min\{l : l \geq 6$ and G_I^{bc} has an induced cycle of length $l\}$, then*

- *$\beta_{i,j} = 0$ if $i < r-4$ and $j > i+3$;*
- *$\beta_{i,j} = 0$ if $i = r-4$ and $j > i+4$;*
- *$\beta_{r-4,r} = \#(\{$induced bipartite cycles in G_I^{bc} of length $r\})$.*

Example. *Let $I = (x_1x_8, x_1x_9, x_2x_9, x_3x_7, x_3x_8, x_4x_6, x_4x_9, x_5x_6, x_5x_8)$*



G_I^{bc}



2 induced 6-cycles in G_I^{bc}

	0	1	2	3	4	5
2	9	11	2	-	-	-
3	-	13	29	19	4	-
4	-	-	2	5	4	1

Betti diagram of I

References

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[2] O. Fernández-Ramos and P. Gimenez, First nonlinear syzygies of ideals associated to graphs. *Communications in Algebra*, **37**(6):1921–1933, 2009.

[3] R. Fröberg, On Stanley-Reisner rings. In: *Topics in Algebra, Part 2* (Warsaw, 1988), Banach Center Publ. **26** (1990) 57–70.