Green's Hyperplane Restriction Theorem for modules

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Abstract

This poster explains a generalization of Green's Hyperplane Restriction Theorem to the case of modules over the polynomial ring, providing in particular an upper bound for the Hilbert function of the general linear restriction of a module M in a degree d by the corresponding Hilbert function of a lexicographic module.

Setting

Definition. *Given $a, d \in \mathbb{N}$, the d-th **Macaulay representation** of a is the only way of writing a as $\binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_1}{1}$, where $a_d > a_{d-1} > \cdots > a_1 \geq 0$.*
Let us set $\binom{c}{d} = 0$ when $c < d$, we denote by $a_{\langle d \rangle}$ the integer

$$a_{\langle d \rangle} = \binom{a_d - 1}{d} + \cdots + \binom{a_1 - 1}{1}.$$

Let $R = k[x_1, \ldots x_n]$, where k is an infinite field, and let R be standard graded. Let us fix the **graded lex** monomial order on R with $x_1 > x_2 > \cdots > x_n$, i.e. $\mathbf{x^a} >_{grlex} \mathbf{x^b} \Leftrightarrow |\mathbf{a}| > |\mathbf{b}|$ or $|\mathbf{a}| = |\mathbf{b}|$ and $\mathbf{a} >_{lex} \mathbf{b}$

Definition. *A monomial ideal $I \subseteq R$ is a **lexicographic ideal** if, for each degree d, I_d is a lex-space, i.e. a vector space generated by the largest $dim(I_d)$ monomials, according to grad-lex.*

Definition. *We say that a property $\mathcal{P}$ holds for a **generic linear form** ℓ if there is a non-empty Zariski open set $\mathcal{U} \subseteq R_1$ such that $\mathcal{P}$ holds for all $\ell \in \mathcal{U}$.*

Green's Theorem

Theorem. *Let I be an homogeneous ideal in S and I_W the restriction to a generic hyperplane, denote by h and h_W respectively the Hilbert functions of S/I and of S_W/I_W, then* $\mathbf{h_W(d) \leq h(d)_{\langle d \rangle}}$.

Extension to modules

Let F be a free finitely generated R-module and let $\{e_1, e_2, \ldots, e_r\}$ be an homogeneous basis, let $\deg(e_i) = f_i$, where, without loss of generality, $f_1 \leq f_2 \leq \cdots \leq f_r$.

Definition. *A monomial in F is an element of the form me_i where $m \in Mon(R)$.*
*A submodule $M \subseteq F$ is **monomial** if it is generated by monomials.*
In this case it can be written as $I_1e_1 \oplus \cdots \oplus I_re_r$, where I_i is a monomial ideal.

The lexicographic order in F is defined as follows: given two monomials in F, me_i and ne_j, we say that $me_i >_{lex} ne_j$ if either $i = j$ and $m >_{lex} n$ in R or $i < j$. In particular, we have that $e_1 > e_2 > \cdots > e_r$.

Definition. *A monomial submodule L is a **lexicographic module** if for every degree d L_d is spanned by the largest, with respect to the lexicographic order, $H(L, d)$ monomials.*

If $\ell \in R_1$ is a linear form, and M a submodule of F, let us denote by $(F/M)_\ell$ the restriction of the module F/M to ℓ, which is equal to $F/(M + \ell F)$.

Green's Theorem for modules

Theorem. *Let $F = Re_1 \oplus \cdots \oplus Re_r$ where $deg(e_i) = f_i$ for all i. Let M be a submodule in F, then*

$$H((F/M)_\ell, d) \leq H((F/L)_\ell, d)$$

where ℓ is generic linear form, $d \in \mathbb{N}$, and L is a submodule that in degree d is generated by a lex-segment of length $H(M, d)$.
Set $d_i = d - f_i$ (it is a non-increasing sequence) and $N_i = \binom{n+d_i-1}{d_i}$. Then:

$$H((F/L)_\ell, d) = H(F/M, d)_{\{d,r\}},$$

where, if $\sum_{i=j+1}^{r} N_i \leq H(F/M, d) \leq \sum_{i=j}^{r} N_i$, for some j, then we define $H(F/M, d)_{\{d,r\}} = (H(F/M, d) - \sum_{i=j+1}^{r} N_i)_{\langle d_j \rangle} + \sum_{i=j+1}^{r} N_{i\langle d_i \rangle}$.

Idea of the proof

We reduce to the monomial case: a monomial submodule is direct sum of monomial components, so we can apply the Green's theorem to each of these components, obtaining a first bound. Afterwards we bound what we get by the Hilbert function of the quotient by lexicographic module. In order to do that we first prove an inequality on the sum of two integers, this inequality is just the numerical translation of the theorem we want to prove in the case rank$(F) = 2$.
Later, we extend the inequality, using an induction argument, to the sum of r integers, and we deduce the extension of the Green's theorem.