HILBERT CURVES OF POLARIZED VARIETIES

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ABSTRACT

Let X be a smooth complex projective variety. We introduce the **Hilbert variety** V_X associated to the Hilbert polynomial, a very classical concept in Algebraic Geometry

$$\chi(x_1L_1 + \ldots + x_\rho L_\rho)$$

where $x_1, \ldots, x_\rho$ are complex variables and $L_1, \ldots, L_\rho$ is a basis of $\mathrm{Pic}(X)$. We study general properties of V_X and we specialize to the **Hilbert curve of the polarized variety** (X, L), namely the plane curve (in the plane $\rangle K_X, L\langle$) of degree $\dim X$, associated to $\chi(xK_X + yL)$. For details we refer to: M. Lavaggi, Invariante j per 3-folds polarizzati, Tesi di Laurea, Corso di Laurea Magistrale, Università degli Studi di Genova, a.a. 2011/2012.

BASIC NOTATION & DEFINITIONS

We work over the field $\mathbb{C}$. We use standard notation from Algebraic Geometry, among which we recall the following ones.
- $\mathcal{O}_X$, the structure sheaf of X.
- K_X is the canonical sheaf of X.
- For any coherent sheaf $\mathcal{F}$ on X, $h^i(\mathcal{F})$ stands for the complex dimension of $H^i(X, \mathcal{F})$.
- $\chi(\mathcal{F}) := \sum_i (-1)^i h^i(\mathcal{F})$, the Euler characteristic of $\mathcal{F}$.

We say that:
- L is *spanned* if it is globally generated, at all points of X by $H^0(X, L)$.
- L is *numerically effective* (*nef*, for short) if $L \cdot C \geq 0$ for all effective curves C on X.
- L is *very ample* if the complete linear system $|L|$ induce an embedding $X \to \mathbb{P}^N$, where $N = h^0(X, L) - 1$.
- We say that L is *ample* if exsists $m > 0$ such that $L^{\otimes m}$ is very ample.

DEGENERATE CASE

Let (X, L) be a polarized variety, and assume that $K_X = \lambda L$ for some $\lambda \in \mathbb{Q}$, so that $\langle K_X, L\rangle$ becomes a line.
Even in this case we can consider the polynomial

$$p(x, y) = \chi(xK_X + yL),$$

defining a plane curve, which we call the *degenerate Hilbert curve*, say Γ_0, of (X, L). Writing $t := \lambda x + y$,

$$p(x, y) = \wp(t) \in \mathbb{C}[t]$$

is a polynomial of degree $n := \dim(X)$ in t and its zeros correspond to the slice $\mathbb{C}_{(t)} \cap V_X$. Moreover, Γ_0 is the union of n parallel lines, ℓ_j, of equation $\lambda x + y - t_j = 0$, where t_j are the roots of $\wp(t)$, $j = 1, \ldots, n$. We refer to this situation as the **"degenerate case"**.

EXAMPLE:
To produce an example, we can consider the polarized variety $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$, clearly $K_X = -3L$.
The Hilbert curve splits in two parallel lines, $r : x - 3y - 2 = 0$ e $t : x - 3y - 1 = 0$. We can also observe that the Hilbert curve is symmetric respect to $C = \left(\frac{1}{2}, 0\right)$.

HILBERT VARIETY: THE GENERAL FRAMEWORK

Here we outline how to obtain the Hilbert variety associated to a smooth n-fold X. The Hilbert variety does not depend by any basis of $\mathrm{Pic}(X)$.
Let $\mathrm{Pic}_0(X) \subset \mathrm{Pic}(X)$ denote the subgroup of topologically trivial line bundles. Set $\mathbf{N}(X) := (\mathrm{Pic}(X)/\mathrm{Pic}_0(X)) \otimes_{\mathbb{Z}} \mathbb{C}$. The Euler characteristic map

$$\chi : \mathrm{Pic}(X) \to \mathbb{Z},$$

defined by $L \mapsto \chi(L)$, gives rise to a polynomial function

$$p : \mathbf{N}(X) \to \mathbb{C}.$$

Note that $\mathbf{N}(X) \cong \mathbb{A}^\rho_{\mathbb{C}}$, where $\rho := \rho(X)$ is the Picard number of X. Via this isomorphism, if $\mathbf{N}(X) = \langle L_1, \ldots, L_\rho\rangle$ with $L_1, \ldots, L_\rho \in \mathrm{Pic}(X)$ and writing $\mathcal{L} = \sum_{i=1}^{\rho} x_i L_i \in \mathbf{N}(X)$, $x_i \in \mathbb{C}$, the image

$$p(\mathcal{L}) = p(x_1, \ldots, x_\rho)$$

is the evaluation in $\mathcal{L}$ of the polynomial $p \in \mathbb{C}[x_1, \ldots, x_\rho]$, when we consider $x_1, \ldots, x_\rho$ as complex variables. In other words, for $x_1, \ldots, x_\rho$ integers, we consider the Hilbert polynomial

$$\chi(x_1, \ldots, x_\rho) := \chi(x_1L_1 + \cdots + x_\rho L_\rho),$$

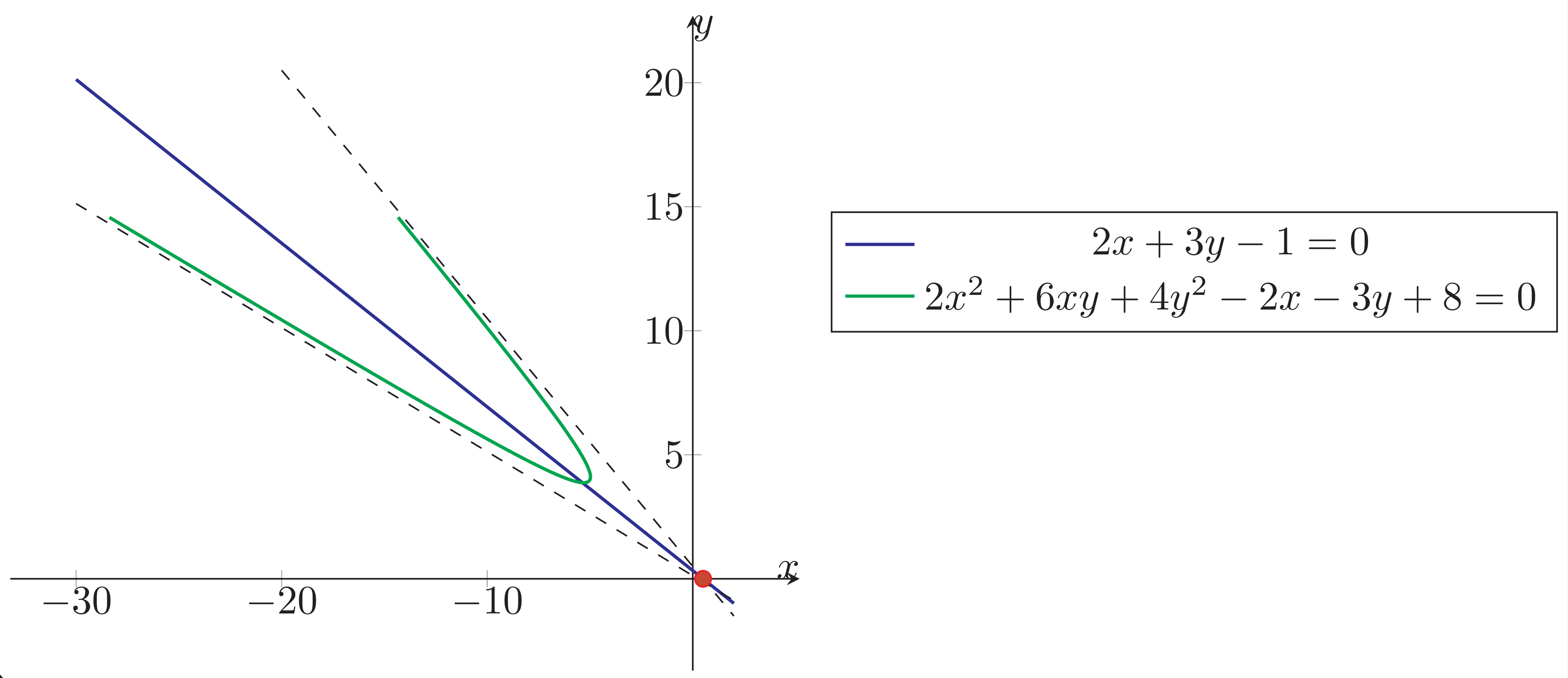
and we denote by $p(x_1, \ldots, x_\rho)$ the polynomial $\chi(x_1, \ldots, x_\rho)$ when we consider $x_1, \ldots, x_\rho$ as complex variables.
Let us consider the affine variety $V_X := V(p)$, which is an hypersurface of degree $\dim(X)$ in $\mathbf{N}(X) \cong \mathbb{A}^\rho_{\mathbb{C}}$. We say that V_X is the **(affine) Hilbert variety associated to X**.

PROPERTIES OF V_X:
- V_X is symmetric with respect to $C = \left(\frac{1}{2}, 0, \ldots, 0\right) \in \mathbb{A}^\rho$;
- For n even, if $C \in V_X$, then V_X is singular at C;
- For any n, if $C \in V_X$ is a point of multiplicity $n - 1$, then C is a point of multiplicity n of V_X.

EXAMPLE

Let X be a smooth element in $|\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(4, 4)|$ and let $L = (\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 2))_X$. Direct computations give for the Hilbert polynomial of the polarized pair (X, L) the expression

$$p(x, y) = (2x + 3y - 1)(2x^2 + 6xy + 4y^2 - 2x - 3y + 8)$$



HILBERT CURVE

Let X be a projective variety of dimension n, let L be ample line bundles on X and consider the Hilbert polynomial

$$\chi(x, y) := \chi(xK_X + yL)$$

with $x, y \in \mathbb{Z}$. Let $p(x, y)$ be the polynomial $\chi(x, y)$ when we consider x, y and z as complex variables. The zeroes of $p(x, y)$ correspond to taking a slice of the Hilbert variety V_X by the 2-dimensional vector subspace $\mathbb{C}^2_{(x,y)} \subseteq \mathbf{N}(X)$ ($\mathbb{C}^2_{(x,y)} = \langle K_X, L\rangle$ whenever K_X, L are $\mathbb{C}$-linearly independent). We will also write

$$V_{(X,L)} := \mathbb{C}^2_{(x,y)} \cap V_X,$$

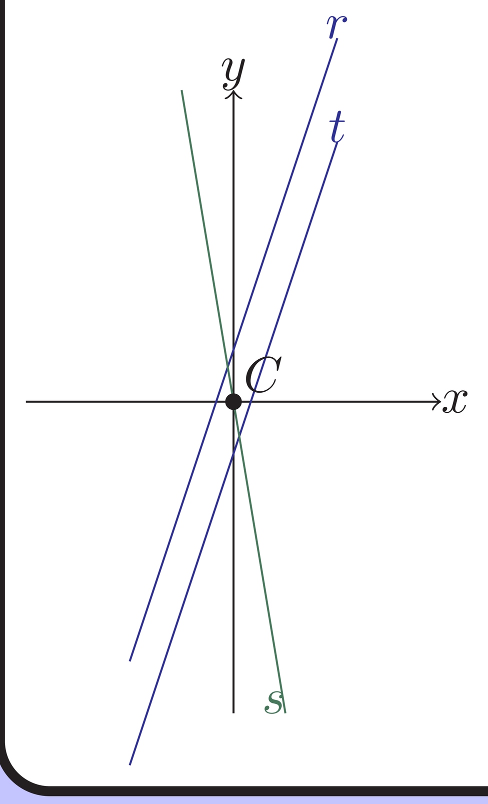
and we will say that the degree $n := \dim(X)$ affine curve $\Gamma := V_{(X,L)}$ is the **Hilbert curve of the polarized variety** (X, L).
For example, if $n = 3$, Γ is an elliptic curve as soon as it is not singular.

HILBERT CURVES FOR SPECIAL VARIETIES

This is a sample of general structure result we have.

Theorem 1 *Let X be a smooth n-dimensional variety, and let $\varphi : X \to Y$ be a morphism onto a normal variety Y of dimension $\dim(Y) < \dim(X)$. Let L be a φ-nef and φ-big line bundle on X, and assume that for coprime positive integers a, b, $K_X + \frac{a}{b}L = \varphi^*A$ for some $\mathbb{Q}$-line bundle A on Y. Then $\chi(xK_X + yL) = 0$ for all integers x, y belonging to the $a - 1$ parallel lines $ax - by - i = 0$ for $i = 1, \ldots, a - 1$. In particular, for some degree $n - a + 1$ factor $R(x, y)$ we have*

$$p(x, y) = \prod_{i=1}^{a-1}(ax - by - i)R(x, y).$$



This figure shows the Hilbert curve associated to a polarized 3-fold (X, L), where X is a scroll over a curve Y of genus $g = 2$. So it follows that

$$K_X + 3L = \varphi^*A$$

for an ample line bundle on Y. It is easy to show that the Hilbert cubic splits in three lines.

TWO OPEN QUESTIONS

- To analyze the Hilbert curve associated to a polarized 4-fold (X, L). In this case we would deal with quartic plane curves.
- To study bipolarized varieties, namely, 3-tuples (X, L_1, L_2) where L_1 and L_2 are ample line bundle on X. In this case we consider the Hilbert surface associated to the bipolarized variety. If X is a 3-fold, then we deal with a cubic surface.