On (Subideal) Border Bases and Their Generalization to Modules

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Notation

- $P = K[x_1, \dots, x_n]$ is the polynomial ring over a field K.
- $\mathbb{T}^n$ is the monoid of all terms $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha_i \in \mathbb{N}$ in P.
- P^r is the free P-module with canonical basis $\{e_1, \dots, e_r\}$.
- $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$ is the monoid of all terms in P^r, i. e. elements of the form $te_k \in P^r$ with $t \in \mathbb{T}^n$ and $k \in \{1, \dots, n\}$.

Order Modules and Their Borders [4]

- A finite set $\mathcal{O} \subseteq \mathbb{T}^n$ is called an **order ideal** if it is closed under forming divisors.
- For an order ideal $\mathcal{O} \subseteq \mathbb{T}^n$, we let the **border** of $\mathcal{O}$ be
$$\partial\mathcal{O} = (x_1 \cdot \mathcal{O} \cup \cdots \cup x_n \cdot \mathcal{O}) \setminus \mathcal{O} \subseteq \mathbb{T}^n.$$
- Let $\mathcal{O}_1, \dots, \mathcal{O}_r \subseteq \mathbb{T}^n$ be order ideals. Then
$$\mathcal{M} = \mathcal{O}_1 \cdot e_1 \cup \cdots \cup \mathcal{O}_r \cdot e_r \subseteq \mathbb{T}^n\langle e_1, \dots, e_r\rangle$$
is called an **order module** and
$$\partial\mathcal{M} = \partial\mathcal{O}_1 \cdot e_1 \cup \cdots \cup \partial\mathcal{O}_r \cdot e_r \subseteq \mathbb{T}^n\langle e_1, \dots, e_r\rangle$$
is called the **border** of $\mathcal{M}$.

Module Border Bases [4]

Let $\mathcal{M} = \{t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}\}$ be an order module with border $\partial\mathcal{M} = \{b_1 e_{\beta_1}, \dots, b_\nu e_{\beta_\nu}\}$ where we have $t_i, b_j \in \mathbb{T}^n$ and $\alpha_i, \beta_j \in \{1, \dots, r\}$.

- A set of vectors $\mathcal{G} = \{\mathcal{G}_1, \dots, \mathcal{G}_\nu\} \subseteq P^r$ is called an **$\mathcal{M}$-module border prebasis** if the vectors have the form
$$\mathcal{G}_j = b_j e_{\beta_j} - \sum_{i=1}^{\mu} c_{ij} t_i e_{\alpha_i}$$
with $c_{ij} \in K$.
- Let $\mathcal{G} = \{\mathcal{G}_1, \dots, \mathcal{G}_\nu\} \subseteq P^r$ be an $\mathcal{M}$-module border prebasis and let $U \subseteq P^r$ be a P-submodule. We call $\mathcal{G}$ an **$\mathcal{M}$-module border basis** of U if $\mathcal{G} \subseteq U$, if the set
$$\overline{\mathcal{M}} = \{t_1 e_{\alpha_1} + U, \dots, t_\mu e_{\alpha_\mu} + U\} \subseteq P^r/U$$
is a K-vector space basis of P^r/U, and if $\#\overline{\mathcal{M}} = \mu$.

Direct Generalizations from Border Bases [4]

- higher borders, an index
- existence and uniqueness
- a division algorithm, normal remainders, normal forms
- characterizations via
  - a special generation property
  - border form modules
  - rewrite rules
  - commuting matrices
  - liftings of border syzygies
  - a Buchberger criterion
- an algorithm for the computation using linear algebra

Schreyer's Theorem for Module Border Bases [1]

Let $\mathcal{G} = \{\mathcal{G}_1, \dots, \mathcal{G}_\nu\} \subseteq P^r$ be an $\mathcal{M}$-module border basis of $\langle\mathcal{G}\rangle$, let $\Lambda \subseteq P^\nu$ be the set of neighbor liftings of $\mathcal{G}$, and let $\{\varepsilon_1, \dots, \varepsilon_\nu\}$ be the canonical basis of P^ν. Then we can explicitly construct a term ordering τ on $\mathbb{T}^n\langle\varepsilon_1, \dots, \varepsilon_\nu\rangle$ such that Λ is a τ-Gröbner basis of $\mathrm{Syz}_P(\mathcal{G}_1, \dots, \mathcal{G}_\nu)$.

Further Notation

- $\mathcal{M} = \{t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}\} \subseteq \mathbb{T}^n\langle e_1, \dots, e_r\rangle$ is an order module with $t_i \in \mathbb{T}^n$ and $\alpha_i \in \{1, \dots, r\}$.
- $\mathcal{G} = \{\mathcal{G}_1, \dots, \mathcal{G}_\nu\} \subseteq P^r$ is an $\mathcal{M}$-module border prebasis.
- $M = \langle m_1, \dots, m_r\rangle$ is a finitely generated P-module.
- $\varphi : P^r \twoheadrightarrow M, e_k \mapsto m_k$ is a P-module epimorphism.

Generalized Module Border Bases [4]

- We call the set $\varphi(\mathcal{G}) \subseteq M$ an **$(\mathcal{M}, \varphi)$-module border prebasis**.
- We call $\varphi(\mathcal{G}) \subseteq M$ an **$(\mathcal{M}, \varphi)$-module border basis** of a P-submodule $U \subseteq M$ if $\varphi(\mathcal{G}) \subseteq U$, if the set
$$\overline{\varphi(\mathcal{M})} = \{t_1 m_{\alpha_1} + U, \dots, t_\mu m_{\alpha_\mu} + U\} \subseteq M/U$$
is a K-vector space basis of M/U, and if $\#\overline{\varphi(\mathcal{M})} = \#\varphi(\mathcal{M})$.

Characterization [4]

Assume that $\varphi|_\mathcal{M}$ is injective. Then the following conditions are equivalent.

- The $(\mathcal{M}, \varphi)$-module border prebasis $\varphi(\mathcal{G}) \subseteq M$ is an $(\mathcal{M}, \varphi)$-module border basis of $\langle\varphi(\mathcal{G})\rangle$.
- The $\mathcal{M}$-module border prebasis $\mathcal{G} \subseteq P^r$ is an $\mathcal{M}$-module border basis of $\langle\mathcal{G}\rangle$ and we have $\ker(\varphi) \subseteq \langle\mathcal{G}\rangle$.

Remarks [4]

- The assumption that $\varphi|_\mathcal{M}$ is injective is crucial, i. e. there are generalized module border bases which cannot be characterized with the above theorem.
- This characterization allows us to define many of the concepts of module border bases, e. g. a division algorithm, for finitely generated P-modules.
- We can use this characterization to compute generalized module border bases if we can computed $\ker(\varphi) \subseteq P^r$.

Applications [4]

- "usual" border bases (see for instance [3, Section 6.4]):
  - $r = 1$, $M = P$, $\varphi = \mathrm{id}_P$
  - can even be considered as module border bases
- subideal border bases (see for instance [2]):
  - $M = \langle m_1, \dots, m_r\rangle \subseteq P$ with polynomials $m_1, \dots, m_r \in P \setminus \{0\}$, $\varphi : P^r \twoheadrightarrow M, e_k \mapsto m_k$
  - can not be considered as module border bases, in general
  - construction as generalized module border bases yields new results, e. g. characterizations and an algorithm for their computation

Bibliography

[1] Martin Kreuzer and Markus Kriegl, **Gröbner bases for syzygy modules of border bases**, 2013 (in preparation).

[2] Martin Kreuzer and Henk Poulisse, **Subideal border bases**, Mathematics of Computation **80** (2011), 1135–1154.

[3] Martin Kreuzer and Lorenzo Robbiano, **Computational Commutative Algebra 2**, Springer, Heidelberg, 2005.

[4] Markus Kriegl, **Module border bases**, 2013 (in preparation).