Differents for a set of points in $\mathbb{P}^n_K$

Le Ngoc Long

Department of Informatics and Mathematics

University of Passau, Germany

General setting

Throughout this poster we let K be an arbitrary field, let $\mathbb{X} = \{p_1, \dots, p_s\}$ be a set of s distinct K-rational points in the projective space $\mathbb{P}^n_K$ such that $\mathbb{X} \cap \mathcal{Z}^+(X_0) = \emptyset$. We equip $P = K[X_0, \dots, X_n]$ with its standard grading $\deg(X_i) = 1$, and let $\mathcal{I}^+(\mathbb{X})$ be the homogeneous vanishing ideal of $\mathbb{X}$ in P. Then $R = P/\mathcal{I}^+(\mathbb{X}) = \bigoplus_{i\geq 0} R_i$ is the homogeneous coordinate ring of $\mathbb{X}$ in $\mathbb{P}^n_K$. The image of X_i in R is denoted by x_i for $i = 0, \dots, n$.

Differents for a set of points in $\mathbb{P}^n_K$

Let $\mathfrak{J}$ denote the kernel of the canonical multiplication map $\mu : R \otimes_{K[x_0]} R \to R, r \otimes r' \mapsto rr'$. The universal derivation of the $K[x_0]$-algebra R is the homogeneous $K[x_0]$-linear map $d_{R/K[x_0]} : R \to \mathfrak{J}/\mathfrak{J}^2$ given by $d_{R/K[x_0]}(r) = r \otimes 1 - 1 \otimes r + \mathfrak{J}^2$, and the module of Kähler differentials of the $K[x_0]$-algebra R is the graded R-module $\Omega^1_{R/K[x_0]} = \mathfrak{J}/\mathfrak{J}^2$. We known that x_0 is not a zerodivisor of R and R is a graded free $K[x_0]$-module of rank s. Let $Q^h(R)$ be the homogeneous quotient ring of R, i.e.

$$Q^h(R) = \{\frac{r}{r'} \mid r, r' \in R, r' \text{ is a homogeneous non-zerodivisor}\}.$$

- The homogeneous ideal $\vartheta_N(R/K[x_0]) = \mu(\mathrm{Ann}_{R\otimes_{K[x_0]}R}(\mathfrak{J})) \subseteq R$ is called the Noether different of R with respect to x_0.
- The first non-zero Fitting ideal of $\Omega^1_{R/K[x_0]}$ is called the Kähler different of R with respect to x_0 and denoted by $\vartheta_K(R/K[x_0])$.
- The image of the homomorphism of graded R-modules

$$\Phi : \mathrm{Hom}_{K[x_0]}(R, K[x_0]) \hookrightarrow \mathrm{Hom}_{K[x_0,x_0^{-1}]}(Q^h(R), K[x_0, x_0^{-1}]) \xrightarrow{\sim} Q^h(R)$$
$$\varphi \mapsto \varphi \otimes \mathrm{id}_{K[x_0,x_0^{-1}]}$$

  is a homogeneous fraction R-ideal $\mathfrak{C}_{R/K[x_0]}$ of $Q^h(R)$, it is called the Dedekind complement module of R with respect to x_0. Its inverse, $\vartheta_D(R/K[x_0]) = \mathfrak{C}^{-1}_{R/K[x_0]} = \{x \in Q^h(R) \mid x \cdot \mathfrak{C}_{R/K[x_0]} \subseteq R\}$, is called the Dedekind different of R with respect to x_0.

We have relations between $\vartheta_K(R/K[x_0])$, $\vartheta_N(R/K[x_0])$, and $\vartheta_D(R/K[x_0])$ as follows.

Theorem 1 (Ernst Kunz). *Let m be the minimal number of generators of $\Omega^1_{R/K[x_0]}$. Then we have*

$$\vartheta_D(R/K[x_0])^m \subseteq \vartheta_K(R/K[x_0]) \subseteq \vartheta_N(R/K[x_0]) = \vartheta_D(R/K[x_0]).$$

Set $\vartheta_{\mathbb{X}}(R/K[x_0]) := \vartheta_N(R/K[x_0]) = \vartheta_D(R/K[x_0])$. We say $\vartheta_{\mathbb{X}}(R/K[x_0])$ the Noether-Dedekind different of R with respect to x_0.

Proposition 2 (Ernst Kunz). *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a complete intersection with $\mathcal{I}^+(\mathbb{X}) = (F_1, \dots, F_n) \subseteq P$, where F_j is a homogeneous polynomial of degree d_j for $j = 1, \dots, n$. Then we have*

- $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0]) = \left(\frac{\partial(F_1,\dots,F_n)}{\partial(x_1,\dots,x_n)}\right)$, *where* $\frac{\partial(F_1,\dots,F_n)}{\partial(x_1,\dots,x_n)}$ *is a non-zerodivisor of R and has degree* $\deg(\frac{\partial(F_1,\dots,F_n)}{\partial(x_1,\dots,x_n)}) = \sum_{j=1}^n d_j - n$.
- $\mathfrak{C}_{R/K[x_0]} = \left\langle \left(\frac{\partial(F_1,\dots,F_n)}{\partial(x_1,\dots,x_n)}\right)^{-1} \right\rangle_R$.

How to compute the differents for a set of points?

In order to compute the differents, we let $\{F_1, \dots, F_r\}$, $r \geq n$, be a homogeneous generating system of $\mathcal{I}^+(\mathbb{X}) \subset P$. Firstly, we observe that the Kähler different $\vartheta_K(R/K[x_0])$ is generated by the n-minors of the Jacobian matrix $\left(\frac{\partial F_j}{\partial x_i}\right)_{\substack{i=1,\dots,n\\ j=1,\dots,r}}$. Thus, it is not hard to write a CoCoA (an ApCoCoA) function to compute $\vartheta_K(R/K[x_0])$. Now we need to find a way to compute the Noether-Dedekind different $\vartheta_{\mathbb{X}}(R/K[x_0])$. We let $Y_1, \dots, Y_n$ be new indeterminates, let $F_i' = F_i(X_0, Y_1, \dots, Y_n)$ for $i = 1, \dots, r$, and let I' be the ideal of $K[X_0, Y_1, \dots, Y_n]$ generated by $\{F_1', \dots, F_r'\}$. We denote the standard graded polynomial ring $K[X_0, X_1, \dots, X_n, Y_1, \dots, Y_n]$ by Q, denote $R[X_1, \dots, X_n]$ by S, and denote the homogeneous ideal in Q generated by $\{F_1', \dots, F_r'\}$ (resp. by $\{F_1, \dots, F_r\}$) by $I'Q$ (resp. $\mathcal{I}^+(\mathbb{X})Q$). Then we have the following commutative diagram

$$\begin{array}{ccccccc} Q & & & \xrightarrow{\quad\psi\quad} & & & P \\ \downarrow & & & & & & \downarrow \\ Q/I'Q & \xrightarrow{\phi} & S = R[X_1,\dots,X_n] & \xrightarrow{\rho} & R \otimes_{K[x_0]} R & \xrightarrow{\mu} & R \end{array} \qquad (1)$$

where ρ is an R-algebra epimorphism given by $\rho(X_i) = 1 \otimes x_i$ for $i = 1, \dots, n$; ϕ is a $K[x_0]$-algebra isomorphism given by $\phi(Y_i + I'Q) = x_i$ for $i = 1, \dots, n$ and $\phi(X_j + I'Q) = X_j$ for $j = 0, \dots, n$; and ψ is a P-algebra epimorphism defined by $\psi(Y_j) = X_j$ for $j = 1, \dots, n$.

Proposition 3. *Let $I_1 := (\mathcal{I}^+(\mathbb{X})Q + I'Q) :_Q ((X_1 - Y_1, \dots, X_n - Y_n) + I'Q) \subseteq Q$. Then the Noether-Dedekind different of R w.r.t x_0 is given by*

$$\begin{aligned}\vartheta_{\mathbb{X}}(R/K[x_0]) &= \{F(x_1, \dots, x_n) \in R \mid F \in \mathcal{I}^+(\mathbb{X})S :_S (X_1 - x_1, \dots, X_n - x_n)\} \\ &= (\mu \circ \rho \circ \phi)(I_1/I'Q) \\ &= \psi(I_1)/\mathcal{I}^+(\mathbb{X}).\end{aligned}$$

Based on Proposition 3, we can compute a system of generators of $\vartheta_{\mathbb{X}}(R/K[x_0])$. Another way to compute $\vartheta_{\mathbb{X}}(R/K[x_0])$, we use the results of M. Kreuzer and S. Beck in the paper *"How to compute the canonical module of a set of points"* to compute the Dedekind complementary module $\mathfrak{C}_{R/K[x_0]}$, and apply the equality $\vartheta_{\mathbb{X}}(R/K[x_0]) = (x_0^{2r_{\mathbb{X}}}) :_R x_0^{2r_{\mathbb{X}}} \mathfrak{C}_{R/K[x_0]}$, where $r_{\mathbb{X}}$ is the regularity index of $\mathrm{HF}_{\mathbb{X}}$ (see later).

Algorithm for computing a minimal homogeneous generating system of $\vartheta_{\mathbb{X}}(R/K[x_0])$

The following algorithm enables us to compute a minimal homogeneous system of generators of the Noether-Dedekind different of R with respect to x_0.

Algorithm 4. *Let $\mathbb{X} = \{p_1, \dots, p_s\}$ be a set of s distinct K-rational points in $\mathbb{P}^n_K$ such that $\mathbb{X} \cap \mathcal{Z}^+(X_0) = \emptyset$, let $\mathcal{I}^+(\mathbb{X})$ be the homogeneous vanishing ideal of $\mathbb{X}$, let $R = P/\mathcal{I}^+(\mathbb{X})$ be the homogeneous coordinate ring of $\mathbb{X}$, and let $<_\sigma$ be a term ordering on $\mathbb{T}^{n+1} = \mathbb{T}(X_0, X_1, \dots, X_n)$. Consider the following instructions.*

1) Compute the reduced $<_\sigma$-Gröbner basis $\{F_1, \dots, F_r\}$ of $\mathcal{I}^+(\mathbb{X})$ by using the Projective Buchberger-Möller Algorithm.

2) Form the polynomial ring $Q = K[X_0, X_1, \dots, X_n, Y_1, \dots, Y_n]$ and equip it with the standard grading $(\deg(X_i) = \deg(Y_j) = 1)$, and form $I'Q = (F_1(X_0, Y_1, \dots, Y_n), \dots, F_r(X_0, Y_1, \dots, Y_n))$.

3) Let $<_{\overline{\sigma}}$ be a term ordering on $\mathbb{T}(X_0, X_1, \dots, X_n, Y_1, \dots, Y_n)$. Compute a $<_{\overline{\sigma}}$-Gröbner basis $\mathcal{H} = \{H_1, \dots, H_t\}$ of the homogeneous colon ideal

$$I_1 = (\mathcal{I}^+(\mathbb{X})Q + I'Q) :_Q ((X_1 - Y_1, \dots, X_n - Y_n) + I'Q).$$

4) Compute a reduced $<_\sigma$-Gröbner basis $\{H_1', \dots, H_{t'}'\}$ of $J_1 = \psi(I_1)$, where ψ is given by (1). Let $m_i = \deg(H_i')$ for $i = 1, \dots, t'$. We sort the set $\{H_1', \dots, H_{t'}'\}$ such that $m_1 \leq \dots \leq m_{t'}$.

5) We denote the image of H_i' in R by h_i for $i = 1, \dots, t'$, and compute the set

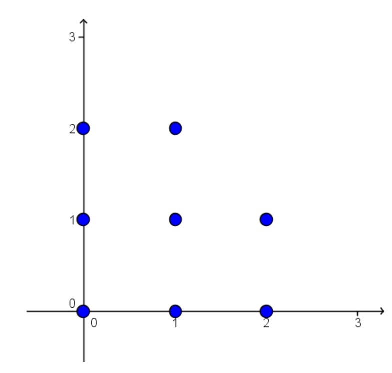
$$\vartheta = \{h_i \mid i \in \{1, \dots, t'\}, H_i' \notin (H_1', \dots, H_{i-1}', F_1, \dots, F_r)\}.$$

6) Return the set $\vartheta \subseteq R$ and stop.

This is an algorithm which computes a minimal homogeneous system of generators of the Noether-Dedekind different $\vartheta_{\mathbb{X}}(R/K[x_0])$ of R with respect to x_0.

Example

Let $\mathbb{X} = \{p_1, \dots, p_8\} \subseteq \mathbb{P}^2_{\mathbb{Q}}$ be a set of 8 distinct points with $p_1 = (1:0:0)$, $p_2 = (1:0:1)$, $p_3 = (1:1:0)$, $p_4 = (1:1:1)$, $p_5 = (1:0:2)$, $p_6 = (1:2:0)$, $p_7 = (1:2:1)$, and $p_8 = (1:1:2)$. Sketch in the affine plane $\mathbb{A}^2_{\mathbb{Q}} = D_+(X_0) = \{p = (c_0 : c_1 : c_2) \in \mathbb{P}^2_{\mathbb{Q}} \mid c_0 \neq 0\}$ as in beside figure. We use $P = \mathbb{Q}[X_0, X_1, X_2]$, $R = P/\mathcal{I}^+(\mathbb{X}) = K[x_0, x_1, x_2]$, and the term ordering DegRevLex. By writing and applying a CoCoA (an ApCoCoA) function which implements Algorithm 4, we calculate $\vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0]) = (x_0^4 - 3x_0x_1^3 + 3/2x_1^4 + 27/4x_0x_1x_2^2 - 9/4x_1^2x_2^2 - 3x_0x_2^3 - 9/4x_1x_2^3 + 3/2x_2^4,\ x_1^2x_2^3 + 5/8x_0x_2^4 - 2x_1x_2^4 + 1/24x_2^5,\ x_1^5 + 429/16x_0x_2^4 - 33x_1x_2^4 - 93/16x_2^5)$.



Hilbert functions of the differents

Let M be a finitely generated graded R-module.

- The Hilbert function of M is defined by $\mathrm{HF}_M(i) = \dim_K(M_i)$ for all $i \in \mathbb{Z}$. In particular, the Hilbert function of R is given by $\mathrm{HF}_{\mathbb{X}}(i) = \dim_K(R_i)$ for $i \in \mathbb{Z}$.
- The regularity index of HF_M is called the regularity index of M and is denoted by r_M. The regularity index of $\mathrm{HF}_{\mathbb{X}}$ will be denoted by $r_{\mathbb{X}}$.

It is well-known that the Hilbert function $\mathrm{HF}_{\mathbb{X}}$ satisfies $\mathrm{HF}_{\mathbb{X}}(i) = 0$ for $i < 0$, $\mathrm{HF}_{\mathbb{X}}(i) = s$ for $i \geq r_{\mathbb{X}}$, and

$$0 < 1 = \mathrm{HF}_{\mathbb{X}}(0) < \mathrm{HF}_{\mathbb{X}}(1) < \cdots < \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}}) = s.$$

The Hilbert functions and the regularity indices of differents are described in the following proposition.

Proposition 5.

- *We have* $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(i) = \mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(i) = 0$ *for $i < 0$ and* $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(i) = \mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(i) = s$ *for* $i \gg 0$.
- *The regularity index of $\vartheta_{\mathbb{X}}(R/K[x_0])$ is exactly $2r_{\mathbb{X}}$.*
- *The regularity index of $\vartheta_K(R/K[x_0])$ satisfies $2r_{\mathbb{X}} \leq r_{\vartheta_K(R/K[x_0])} \leq (n+1)r_{\mathbb{X}}$.*
- *If $\mathbb{X}$ is arithmetically Gorenstein (i.e. R is a Gorenstein ring), then* $\mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(i) = \mathrm{HF}_{\mathbb{X}}(i - r_{\mathbb{X}})$ *for all $i \in \mathbb{Z}$.*
- *If $\mathbb{X}$ is a complete intersection, then* $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(i) = \mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(i) = \mathrm{HF}_{\mathbb{X}}(i - r_{\mathbb{X}})$ *for all $i \in \mathbb{Z}$.*

Differents for Cayley-Bacharach schemes

A set of s distinct K-rational points $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}^n_K$ is called a Cayley-Bacharach scheme, if every hypersurface of degree less than $r_{\mathbb{X}}$ which contains all but one point of $\mathbb{X}$ must contain all the points of $\mathbb{X}$. The Hilbert functions of differents for a Cayley-Bacharach scheme are described as follows.

Proposition 6. *If $\mathbb{X} \subseteq \mathbb{P}^n_K$ is a Cayley-Bacharach scheme, then for every $i \in \mathbb{Z}$ we have*

$$\mathrm{HF}_{\vartheta_K(R/K[x_0])}(i) \leq \mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(i) \leq \mathrm{HF}_{\mathbb{X}}(i - r_{\mathbb{X}}).$$

In particular, the Hilbert function of $\vartheta_{\mathbb{X}}(R/K[x_0])$ satisfies $\mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(i) = 0$ for $i < r_{\mathbb{X}}$, $\mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(i) = s$ for $i \geq 2r_{\mathbb{X}}$, and

$$0 \leq \mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(r_{\mathbb{X}}) \leq \cdots \leq \mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(2r_{\mathbb{X}} - 1) < \mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(2r_{\mathbb{X}}) = s.$$

Our next proposition gives criteria for a set of s distinct K-rational points in $\mathbb{P}^n_K$ to be either an arithmetically Gorenstein scheme or a complete intersection.

Proposition 7. *Let $\mathbb{X}$ be a set of s distinct K-rational points in $\mathbb{P}^n_K$.*

- *$\mathbb{X}$ is arithmetically Gorenstein if and only if it is a Cayley-Bacharach scheme and* $\mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(r_{\mathbb{X}}) \neq 0$.
- *$\mathbb{X}$ is a complete intersection if and only if it is a Cayley-Bacharach scheme and* $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(r_{\mathbb{X}}) \neq 0$.

The International School on Computer Algebra: COCOA 2013 (10-14th June 2013) at Institute of Mathematics, University of Osnabrück, Germany.