SCHUBERT3–A SAGE package for Intersection Theory and Enumerative Geometry

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1. Introduction

- SCHUBERT–A MAPLE package was written by S. Katz and S. A. Strømme from 1992, and updated to new versions of MAPLE by J.-M. Økland. However, it is no longer actively supported.
- SCHUBERT2 in MACAULAY2 has been developed by D. R. Grayson, M. E. Stillman, S. A. Strømme, D. Eisenbud and C. Crissman.
- Our package is SCHUBERT3 which developed on SAGE and written by PYTHON programming language. This package supports the computation in Intersection Theory on smooth varieties. It deals with varieties, vector bundles on varieties, equivariant vector bundles on varieties endowed with torus actions and morphisms between varieties.

2. SAGE Code

In order to use SCHUBERT3, we first must attach the file `Schubert3.py` to a running session using the `%attach` command in SAGE. This file is available from the author upon request. The current version of this package is the result of many discussions on mathematical background and on implementation algorithms, and of many hours of coding. It should not be considered a complete, unchangeable, totally stable version. We will keep working on this project by fixing bugs, improving algorithms, and by adding functionalities.

3. Varieties and Chow Rings

In SCHUBERT3, a variety is given by the dimenion, variables, degrees, and relations such that we can determine its Chow ring.

3.1 Projective Spaces

An n-dimensional projective space $\mathbb{P}^n$ is given by dimension n, one variable h with $\deg(h) = 1$ and one relation h^{n+1}. Thus the Chow ring of $\mathbb{P}^n$ is

$$A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1}),$$

where h is the hyperplane class of $\mathbb{P}^n$.

3.2 Grassmannians

A Grassmannian $G(k,n)$ is given by dimension $k(n-k)$, $n-k$ variables $\sigma_1,\dots,\sigma_{n-k}$ with $\deg(\sigma_i) = i$ and $n-k$ relations $\sigma_{1^i}, i = k+1,\dots,n$, where σ_i and σ_{1^i} are the special Schubert classes on this Grassmannian. Thus its Chow ring has the form

$$A^*(G(k,n)) \cong \frac{\mathbb{Z}[\sigma_1,\dots,\sigma_{n-k}]}{I},$$

where the ideal I is generated by the $\sigma_{1^i}, i = k+1,\dots,n$. Moreover, this ring is graded with $\deg(\sigma_i) = i$.

Algorithm 1: *Integration on Grassmannians*

Input: The zero-dimensional cycle class α on the Grassmannian $G(k,n)$. In fact, α is a homogeneous polynomial of degree $k(n-k)$ in $\mathbb{Z}[\sigma_1,\dots,\sigma_{n-k}]$.
Output: The degree of cycle class α in the Chow ring of $G(k,n)$, denoted by

$$\int_{G(k,n)} \alpha.$$

1. Using Giambelli's formula for computing the ideal I generated by

$$\sigma_{1^i}, i = k+1,\dots,n.$$

2. Reduce α modulo I, store f.
3. **Return** the leading coefficient of f.

4. Lines on Hypersurfaces

We want to compute the number of the lines on a general hypersurface of degree $d = 2n-3 \geqslant 3$ in $\mathbb{P}^n$. This number is the degree of the top Chern class of the d-th symmetric power of the tautological quotient bundle on $G(n-1,n+1)$. For example, there are 27 lines on a general cubic hypersurface in $\mathbb{P}^3$ and 2875 lines on a general quintic threefold in $\mathbb{P}^4$.

5. Vector Bundles and Chern Classes

A vector bundle on a variety is given by the rank and the Chern classes or the Chern character. The idea for implementation of vector bundles is based on the splitting principle.
An equivariant vector bundle on a variety endowed with the torus action is given by its weights of the torus action on the ordinary vector bundle.

5.1 Splitting Principle

The splitting principle is a useful technique used to reduce questions about vector bundles to the case of line bundles. The splitting principle is usually formalized as follows. Let E be a vector bundle of rank r. We can regard the Chern classes of E as the elementary symmetric polynomials in r variables $\alpha_1,\dots,\alpha_r$ called the *Chern roots* of E. More precisely, we can write formally the total Chern class of E as follows:

$$c(E) = \prod_{i=1}^{r}(1+\alpha_i).$$

Equivalently, we have

$$c_0(E) = 1, c_1(E) = \sum_{1\leqslant i\leqslant r}\alpha_i, c_2(E) = \sum_{1\leqslant i<j\leqslant r}\alpha_i\alpha_j, \dots, c_r(E) = \alpha_1\alpha_2\dots\alpha_r.$$

Let E and F be two vector bundles on a variety X with Chern roots $(\alpha_i)_i$ and $(\beta_j)_j$, respectively. Then we have the following statements.

- $E^\vee$ has Chern roots $(-\alpha_i)_i$.
- $E \oplus F$ has Chern roots $(\alpha_i, \beta_j)_{i,j}$.
- $E \otimes F$ has Chern roots $(\alpha_i + \beta_j)_{i,j}$.
- $\mathrm{Sym}^d E$ has Chern roots $(\alpha_{i_1} + \dots + \alpha_{i_d})_{i_1\leqslant\dots\leqslant i_d}$.
- $\wedge^d E$ has Chern roots $(\alpha_{i_1} + \dots + \alpha_{i_d})_{i_1<\dots<i_d}$.

These give us the useful tools for implementing almost standard operations of the vector bundles.

5.2 Chern Characters and Todd Classes

Using the notion of Chern roots, we can define formally the *Chern character* and the *Todd class* of a vector bundle E as follows:

$$\mathrm{ch}(E) = \sum_{i=1}^{r}\exp(\alpha_i) \quad , \quad \mathrm{td}(E) = \prod_{i=1}^{r}\frac{\alpha_i}{1-\exp(-\alpha_i)},$$

where $\alpha_1,\dots,\alpha_r$ are the Chern roots of E and the expressions in the α_i are understood as formal power series, i.e.

$$\exp(\alpha_i) = 1 + \alpha_i + \frac{1}{2}\alpha_i^2 + \frac{1}{6}\alpha_i^3 + \cdots \quad , \quad \frac{\alpha_i}{1-\exp(-\alpha_i)} = 1 + \frac{1}{2}\alpha_i + \frac{1}{12}\alpha_i^2 + \cdots.$$

Theorem 5.1 (Hirzebruch-Riemann-Roch). *Let E be a vector bundle on a smooth, complete variety X. Then we have the following formula*

$$\chi(X,E) = \int_X \mathrm{ch}(E)\cdot\mathrm{td}(X),$$

where $\chi(X,E)$ denotes the Euler characteristic of E on X and $\mathrm{td}(X)$ denotes the Todd class of the tangent bundle on X.

6. Chow Rings of Projective Bundles

Let E be a vector bundle of rank $r+1$ on a smooth projective scheme X, and let $\mathbb{P}(E)$ be a projective bundle of E on X. If $p : \mathbb{P}(E) \to X$ is the projection morphism, then the pullback $p^* : A(X) \to A(\mathbb{P}(E))$ is an injection of rings. Moreover, the Chow ring of $\mathbb{P}(E)$ has the form

$$A(\mathbb{P}(E)) \cong \frac{A(X)[\zeta]}{(\zeta^{r+1} + c_1(E)\zeta^r + \dots + c_r(E)\zeta + c_{r+1}(E))},$$

where $\zeta = c_1(S^\vee) \in A^1(\mathbb{P}(E))$, and $S^\vee$ is the dual of the tautological line bundle on $\mathbb{P}(E)$.

7. Conics on Quintic Threefolds

Let X be a general quintic threefold in $\mathbb{P}^4$. How many rational curves of degree d are contained in X? This is a difficult question in enumerative geometry and still unresolved. In case, $d = 1$, it was known in Section 4 that there are 2875 lines on X. In this section we will set up the computation of the number of degree 2 rational curves (smooth conics) on a general quintic threefold. This number was computed by S. Katz in 1985. Here is how the computation can be made with SCHUBERT3.
Let $G(3,5)$ be the Grassmannian of planes in $\mathbb{P}^4$, with tautological subbundle S. The space of smooth conics in $\mathbb{P}^4$ may be identified with the projective bundle $\mathbb{P}(\mathrm{Sym}^2(S^\vee))$ over $G(3,5)$.

```
sage: G = Grassmannian(3,5)
sage: S = G.tautological_subbundle().dual()
sage: B = S.symmetric_power(2)
sage: PB = ProjectiveBundle(B)
```

The cycle class of smooth conics on a general quintic threefold is the top Chern class of quotient vector bundle

$$A = \mathrm{Sym}^5(S^\vee) - \mathrm{Sym}^3(S^\vee) \otimes \mathcal{O}_{\mathbb{P}(\mathrm{Sym}^2(S^\vee))}(-1).$$

```
sage: V = PB.O(-1) & S.symmetric_power(3)
sage: A = S.symmetric_power(5) - V
sage: C = A.top_chern_class()
```

The number of smooth conics on a general quintic threefold is equal to the degree of this cycle class in the Chow ring $A^*(\mathbb{P}(\mathrm{Sym}^2(S^\vee)))$,

$$\int_{\mathbb{P}(\mathrm{Sym}^2(S^\vee))} c_{11}(A).$$

```
sage: PB.integral(C)
609250
```

If $p : \mathbb{P}(\mathrm{Sym}^2(S^\vee)) \to G(3,5)$ is the projection morphism, then we have

$$\int_{\mathbb{P}(\mathrm{Sym}^2(S^\vee))} c_{11}(A) = \int_{G(3,5)} p_*(c_{11}(A)).$$

This means that the number of smooth conics on a general quintic threefold is also equal to the degree of $p_*(c_{11}(A))$ in the Chow ring $A^*(G(3,5))$.

```
sage: p = PB.projection_morphism()
sage: G.integral(p.pushforward(C))
609250
```

8. Further Works

- Excess intersection formula, chow rings of blowups.
- Bott's formula and Gromov-Witten invariants.
- SINGULAR code, available at the homepage of SINGULAR.
- CoCoA???