Persistence and Stability Properties of Powers of Ideals

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(Based on joint work with Jürgen Herzog)

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1 Abstract

We introduce the concept of strong persistence and show that it implies persistence regarding the associated prime ideals of the powers of ideal. We also show that strong persistence is equivalent to a condition on power of ideals studied by Ratliff. Furthermore, we give an upper bound for the depth of powers of monomial ideals in terms of their linear relation graph, and apply this to show that the index of depth stability and the index of stability for the associated prime ideals of polymatroidal ideals is bounded by their analytic spread.

2 Algebraic background and history

Let I be an ideal in a Noetherian ring R. The story begins with a question of Ratliff who in the 70'th asked:

What happens to $\mathrm{Ass}(I^n)$ as n gets large?

It is a general phenomenon that algebraic and homological properties of I^n stabilize for large n.

Brodmann in 1979 showed that $\mathrm{Ass}(I^n)$ stabiles for large n. The smallest integer for which $\mathrm{Ass}(I^n)$ stabilizes is called the index of stability of I and denoted by $\mathrm{astab}(I)$.

The behavior of the $\mathrm{depth}(I^n)$ is strongly related to $\mathrm{Ass}(I^n)$. For a graded ideal I in a polynomial ring S, the function $f(k) = \mathrm{depth}(S/I^k)$ is called the depth function of I^k. This function is first studied in [6]. It is known that $\mathrm{depth}(S/I^k)$ is constant for $k >> 0$. The smallest integer for which $\mathrm{depth}(I^k)$ stabilizes is called the index of stability of I and denoted by $\mathrm{dstab}(I)$.

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3 Definitions

(1) An ideal I is said to satisfy the persistence property if $\mathrm{Ass}(I) \subset \mathrm{Ass}(I^2) \subset \cdots \subset \mathrm{Ass}(I^k) \subset \cdots$.

(2) Let P be a prime ideal containing I. We say that I satisfies the strong persistence property with respect to P if for all k and all $f \in (I_P^k : \mathfrak{m}_P) \setminus I_P^k$ there exists $g \in I_P$ such that $fg \notin I_P^{k+1}$. The ideal I is said to satisfy the strong persistence property if it satisfies the strong persistence property for all P containing I.

(3) An ideal I is said to satisfy Ratliff condition if $I^{k+1} : I = I^k$, for all k. Ratliff showed in [8] that this condition is satisfied for any normal ideal, and that $I^{k+1} : I = I^k$ for all $k \gg 0$, in general.

Theorem 3.1. *The ideal $I \subset R$ satisfies the strong persistence property if and only if $I^{k+1} : I = I^k$ for all k.*

4 Polymatroidal Ideals

The set of bases of a polymatroid of rank d based on $[n]$ is a set $\mathcal{B} \subset \mathbb{Z}^n$ of integer vectors $\mathbf{a} = (a(1), \dots, a(n))$ with non-negative entries satisfying the following conditions:

(i) $|\mathbf{a}| = \sum_{i=1}^n a(i) = d$ for all $\mathbf{a} \in \mathcal{B}$;

(ii) (Exchange property) For all $\mathbf{a}, \mathbf{b} \in \mathcal{B}$ for which $a(i) > b(i)$ for some i, there exists $j \in [n]$ such that $b(j) > a(j)$ and $\mathbf{a} - \epsilon_i + \epsilon_j \in \mathcal{B}$. Here ϵ_i denotes the canonical ith unit vector.

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Definition 4.1. A monomial ideal $I \subset S = K[x_1, \dots, x_n]$ is called a polymatroidal ideal, if there exists a set of bases $\mathcal{B} \subset \mathbb{Z}^n$ of a polymatroid, such that

$$G(I) = \{\mathbf{x}^{\mathbf{a}} \;\; \mathbf{a} \in \mathcal{B}\}.$$

Here we denote by $G(I)$ the unique minimal set of generators of a monomial ideal I.

Proposition 4.2. *Let I be a polymatroidal ideal. Then I satisfies the strong persistence property.*

5 Main Result

Theorem 5.1. *Let $I \subset S = K[x_1, \dots, x_n]$ be a polymatroidal ideal. Then*

$$\mathrm{astab}(I), \mathrm{dstab}(I) < \ell(I)$$

. In particular, $\mathrm{astab}(I), \mathrm{dstab}(I) < n$.

To prove the above theorem, we first introduce the linear relation graph γ associated to an ideal I in a polynomial ring $S = K[x_1, \dots, x_n]$.

Definition 5.2. Let $G(I) = \{u_1, \dots, u_m\}$. The linear relation graph Γ of I is the graph with edge set

$$E(\Gamma) = \{\{i, j\} \text{ there exist } u_k, u_l \in G(I) \text{ such that } x_i u_k = x_j u_l\}$$

and vertex set $V(\Gamma) = \bigcup_{\{i,j\} \in E(\Gamma)} \{i, j\}$.

Example 5.3. Let I_G be the edge ideal of the finite simple graph G on the vertex set $[n]$. Then the linear relation graph Γ of I_G has edge set

$$\{\{i, j\} \; : i, j \in V(G) \text{ and } i \text{ and } j \text{ have a common neighbor in } G\}.$$

We also need the following result to prove Theorem 5.1.

Theorem 5.4. *Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal generated in a single degree whose linear relation graph Γ has r vertices and s connected components. Then*

$$\mathrm{depth}\, S/I^t \leq n - t - 1 \quad \textit{for } t = 1, \dots, r - s.$$

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Lemma 5.5. *Let I be a monomial ideal and Γ be the linear relation graph of I. Suppose Γ has r vertices and s connected components. Then*

$$\ell(I) \geq r - s + 1,$$

and equality holds if I is a polymatroidal ideal.

Conjecture 5.6. In general the indices $\mathrm{astab}(I)$ and $\mathrm{dstab}(I)$ are unrelated, as shown by examples given in [4]. On the other hand on the evidence of all known examples we conjecture that $\mathrm{astab}(I) = \mathrm{dstab}(I)$ for all polymatroidal ideals I.

References

[1] M. Brodmann, Asymptotic stability of $\mathrm{Ass}(M/I^nM)$, Proc. AMS **74** (1979), 16-18 .

[2] M. Brodmann, The asymptotic nature of the analytic spread, Math. Proc. Cambridge Philos. Soc. , **86** (1979), 35-39.

[3] CoCoA Team, CoCoA: a system for doing computations in commutative algebra, Available at http: //cocoa.dima.unige.it.

[4] J. Herzog, A. Rauf, M. Vladoiu, The stable set of associated prime ideals of a polymatroidal ideal, J. Algebraic Combinatorics, DOI 10.1007/s10801-012-0367-z.

[5] J. Herzog, T. Hibi, *Monomial ideals*. Graduate Texts in Mathematics, 260. Springer-Verlag London, Ltd., London, 2011.

[6] J. Herzog, T. Hibi, The depth of powers of an ideal, J. Algebra **291** (2005), 534-550.

[7] J. Herzog, T. Hibi, Discrete polymatroids, J. Algebraic Combinatorics **16** (2002), 239-268.

[8] L. J. Ratliff Jr. On prime divisors of I^n, n large, Michigan Math. J., **23** (1976), 337-352.

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