On the Arithmetical Rank of certain Monomial Ideals

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Abstract We consider a large class of squarefree monomial ideals, generated in any degree, attached to simplicial complexes and we prove that, for these ideals, the arithmetical rank is equal to the big height. In the case of squarefree monomial ideals generated only in degree two, the same equality can be proven for a larger class of ideals.

1. Bounds for the Arithmetical Rank

Suppose that R is a Noetherian commutative ring with identity.

Definition Let R be a ring and $I \subset R$ an ideal. The **arithmetical rank** of I, denoted by ara I, is the smallest integer s for which there exist s elements $a_1, \ldots, a_s$ of R such that $\sqrt{I} = \sqrt{(a_1, \ldots, a_s)}$.

Let k be a field, $R = k[x_1, \ldots, x_n]$ and I a squarefree monomial ideal in R. Then the following inequalities hold:

$$\text{ht}\, I \leq \text{bight}\, I \leq \text{pd}_R(R/I) = \text{cd}\, I \leq \text{ara}\, I \leq \mu(I),$$

where bight I (*big height*) is the maximum height of the minimal prime ideals of I, pd is the *projective dimension* and cd I is the *cohomological dimension* of I, that is

$$\text{cd}\, I = \max\{i \in \mathbb{Z} : H^i_I(R) \neq 0\}.$$

The second inequality follows from the graded Auslander-Buchsbaum formula and equality holds if and only if R/I is a Cohen-Macaulay ring. The equality between $\text{pd}_R(R/I)$ and cd I was proven by Lyubeznik (1984). Finally, the third inequality follows from a result due to Hartshorne (1968). A trivial upper bound for the arithmetical rank is the minimum number of generators of I, denoted by $\mu(I)$. This is the length of the *Taylor resolution* of I. In general, all these inequalities are strict.

Definition An ideal I of R is called a **set-theoretic complete intersection** if ara I = ht I.

Corollary *Let $R = k[x_1, \ldots, x_n]$ and I a squarefree monomial ideal. If I is a set-theoretic complete intersection, then* ara $I = \text{pd}_R(R/I) = $ ht I. *Hence, R/I is a Cohen-Macaulay ring.*

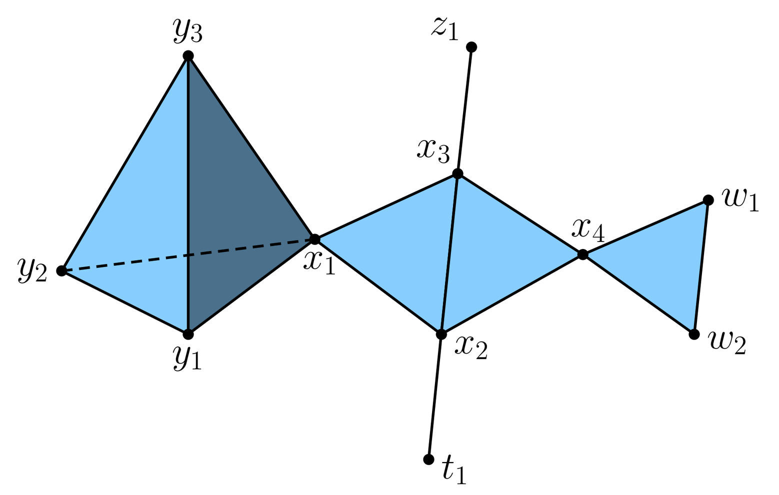
2. Facet Ideals

Definition Let Δ be a simplicial complex on the vertex set $\{x_1, \ldots, x_n\}$. The **facet ideal** of Δ is the ideal $I(\Delta)$ in $k[x_1, \ldots, x_n]$ generated by all squarefree monomials $x_{i_1} \cdots x_{i_s}$, where $\{x_{i_1}, \ldots, x_{i_s}\}$ is a facet of Δ.

Theorem (Faridi, 2005) *Let Δ be a simplicial complex on the vertex set $\{x_1, \ldots, x_n\}$. For every vertex x_i, we add a facet F_i of dimension ≥ 1 such that*

- *$F_i \cap \Delta = \{x_i\}$, for every $i = 1, \ldots, n$,*
- *$F_i \cap F_j = \varnothing$ if $i \neq j$, for every $i, j = 1, \ldots, n$.*

Call Δ' the simplicial complex obtained in this way. Then the facet ideal $I(\Delta')$ is Cohen-Macaulay and ht $I(\Delta') = n$.



Theorem (M.) *Let Δ be a simplicial complex on the vertex set $\{x_1, \ldots, x_n\}$. For every vertex x_i, we add $m_i \geq 1$ facets $F_{i,1}, \ldots, F_{i,m_i}$ of dimension ≥ 1 such that*

- *$F_{i,j} \cap \Delta = \{x_i\}$, for every $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$,*
- *$F_{i,j} \cap F_{i,k} = \{x_i\}$, for every $i = 1, \ldots, n$ and $j, k = 1, \ldots, m_i$,*
- *$F_{i,j} \cap F_{h,k} = \varnothing$, if $i \neq h$, for every $i, h = 1, \ldots, n$, $j = 1, \ldots, m_i$ and $k = 1, \ldots, m_k$.*

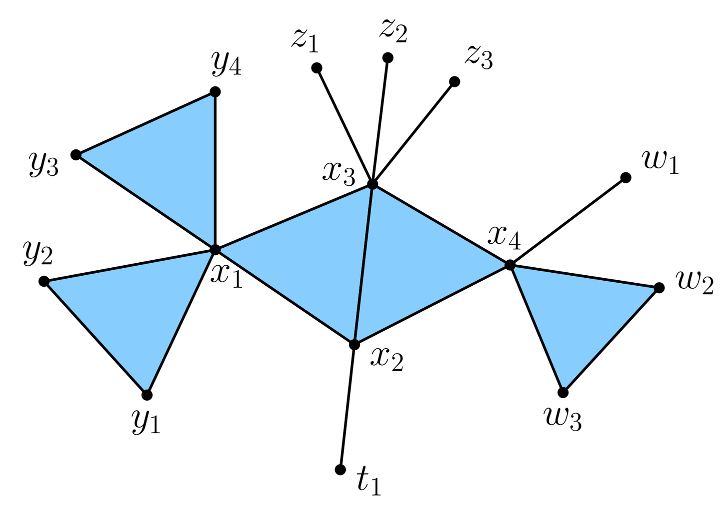
Call Δ' the simplicial complex obtained in this way. Then

$$\text{ara}\, I(\Delta') = \text{bight}\, I(\Delta').$$

In particular, if $m_i = 1$ for every $i = 1, \ldots, n$, then

$$\text{ara}\, I(\Delta') = \text{ht}\, I(\Delta') = n,$$

thus $I(\Delta')$ is a set-theoretic complete intersection.



3. Edge Ideals

Now we consider the case of the edge ideals, that are squarefree monomial ideals generated in degree two.

Let G be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$, for some integer $n \geq 1$, and whose edge set is $E(G)$. Suppose that $x_1, \ldots, x_n$ are indeterminates over a field k.

Definition The **edge ideal** of G in the polynomial ring $R = k[x_1, \ldots, x_n]$ is the squarefree monomial ideal

$$I(G) = (\{x_i x_j : \{x_i, x_j\} \in E(G)\}).$$

For the sake of simplicity, we will use the same notation $x_i x_j$ for the monomial and for the corresponding edge.

Definition A graph G is called **Cohen-Macaulay** on the field k if $R/I(G)$ is a Cohen-Macaulay ring.

In many cases it has been proven that ara $I = \text{pd}_R(R/I)$. For example, this equality has been established for lexsegment edge ideals (by Ene, Olteanu, Terai, 2010), for the graphs formed by one or two cycles connected through a path, the so-called *cyclic* and *bicyclic* graphs (by Barile, Kiani, Mohammadi and Yassemi, 2012) and for the graphs consisting of paths and cycles with a common vertex (by Kiani and Mohammadi, 2012).

A stronger condition is the equality between the arithmetical rank and the big height. This is the case for the edge ideals of acyclic graphs, the so-called *forests*.

Theorem (Kimura, Terai, 2013) *Let T be a forest. Then* ara $I(T)$ = bight $I(T)$.

In all these cases, the arithmetical rank is independent of the field k.

Definition Let G be a graph with vertex set $\{x_1, \ldots, x_n\}$. Adding a **whisker** to the vertex x_i means adding a new vertex y_i and the edge connecting x_i and y_i.

Definition A subset C of $V(G)$ is a **clique** if it induces a complete subgraph of G. A **clique vertex-partition** of G is a set $\pi = \{W_1, \ldots, W_t\}$ of disjoint (possibly empty) cliques of G whose union is $V(G)$.

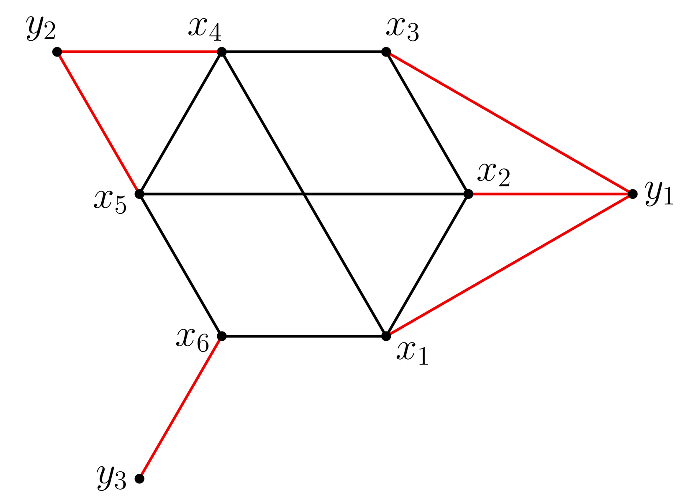
Definition Given a clique W of G, a **clique-whiskering** of W is given by adding a new vertex w and connecting w to every vertex in W. Let $\pi = \{W_1, \ldots, W_t\}$ be a clique vertex-partition of G. Consider the clique-whiskering of every clique of π obtained by adding the vertex w_i to W_i (where $w_i \neq w_j$ if $i \neq j$). We call the graph G^π obtained in this way **fully clique-whiskered**. This graph has vertex set $V(G) \cup \{w_1, \ldots, w_t\}$ and edge set $E(G) \cup \{vw_i \mid v \in W_i\}$.

Theorem (Cook, Nagel, 2012) *Let π be a clique vertex-partition of a graph G and let G^π be the fully clique-whiskering graph of G on π. Then the ideal $I(G^\pi)$ is Cohen-Macaulay and* ht $I(G^\pi) = n$.

Proposition (M.) *Let G be a graph on the vertex set $V(G) = \{x_1, \ldots, x_n\}$. Consider a partition $\{W_1, \ldots, W_t\}$ of $V(G)$. For all $i = 1, \ldots, t$, and for every $x_j \in W_i$ add a new vertex y_i and the whisker $x_j y_i$. Let G' be the graph obtained in this way. Then*

$$\text{ara}\, I(G') = \text{bight}\, I(G') = n.$$

In particular, every fully clique-whiskered graph is a set-theoretic complete intersection.

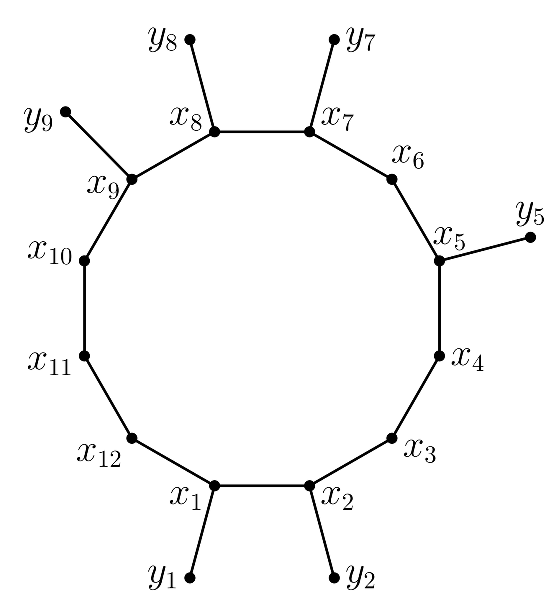


4. Cycles with Missing Whiskers

The equality between the arithmetical rank and the big height holds also in the following case:

Proposition (M.) *Let G be a graph obtained by adding a whisker to some of the vertices of a cycle graph C_n on n vertices. Then*

$$\text{ara}\, I(G) = \text{bight}\, I(G).$$



In most cases, we can prove this equality subdividing the graph G in two subgraphs H and K that are trees. From the result due to Kimura and Terai, the arithmetical rank is equal to big height, both for $I(H)$ and $I(K)$. Then, we prove that ara $I(G)$ = ara $I(H)$+ara $I(K)$ = bight $I(H)$+bight $I(K)$ = bight $I(G)$. In the remaining cases, this technique does not provide the equality between the arithmetical rank and the big height. In order to prove the claim, we define ad hoc polynomials that generate $I(G)$ up to radical.

5. Conclusions and Future Works

Now we are studying a more general situation: we consider the edge ideal $I(G)$ of a graph with an arbitrary number of pairwise disjoint cycles and we want to find an upper bound for the difference between $\text{pd}_R(R/I(G))$ and bight $I(G)$. It could be that, for example, this difference is bounded above by the number of cycles of G. This is an extension of the result due to Kimura and Terai, according to which for a forest T (a graph without cycles), $\text{pd}_R(R/I(T))$ = bight $I(T)$.