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Stanley Conjecture for squarefree monomial ideals

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Introduction

Richard Stanley in his article "Linear Diophantine equations and local cohomology", Invent. Math. 68 (1982) [23] made a striking conjecture predicting an upper bound for the depth of a multigraded module. This conjectured upper bound is nowdays called the Stanley depth of a module. The Stanley depth is of a rather combinatorial nature while the depth is a homological ivnariant. The conjecture is on so far striking as it compares two invariants of modules of very different nature. At a first glance it seems to be no relation among these two invariants.

Here we concentrate on the case that $M = I/J$ is a finitely generated Z^n- graded S-module, where $S = K[x_1, ..., x_n]$ is the polynomial ring and $J \subset I \subset S$ are monomial ideals generated by squarefree monomials of degrees $\geq d$, resp. $\geq d+1$ for some $d \in N$.

Known cases

For more than thirty years the Stanley Conjecture was a dream for many people working in combinatorics and commutative algebra. Many people believe that this conjecture holds and tried to prove directly some of its consequences. For example in this way a lower bound of depth given by Lubeznik [12] was extended by Herzog at al. [6] for sdepth. Here are some cases when the conjecture is verified:

- All algebras S/I, I a monomial ideal, if it holds for all such Cohen Macaulay algebras [7].
- If Δ is simplicial complex. Then the Stanley Reisner ring $K[\Delta]$ of Δ satisfies Stanleys conjecture, if is shellable
- S/I when I is Cohen-Macaulay ideal of codimension 2 or Gorenstein of codimension 3. [8].
- S is a polynomial ring in at most 5 variables, then Stanleys conjecture holds for I and S/I [14].
- M is an almost clean module [5] a monomial ideal I such that this I is the intersection of four prime monomial ideals [13], [15], or the intersection of three monomial primary ideals [25], or an almost complete intersection ideal [3].
- $M = I/J$, sdepth $I/J = d+1$ and I is generated by at most one generator of degree d and some others generators of degree > d [17], [19].

Open questions

Let $I \subset S$ be a monomial ideal. Is it true that sdepth $(I^k) \geq$ sdepth (I^{k+1}) for all k? In general such an inequality is not true for the ordinary depth, however it is conjectured that depth $(I^k) \geq$ depth (I^{k+1}) when I is a squarefree monomial ideal.

Let $I \subset S$ be a monmial ideal. Then sdepth$(I) \geq$ sdepth (S/I). In all known cases one even has sdepth $(I) >$ sdepth (S/I).

Let $J \subset I \subset S$ be squarefree monomial ideals. If all generators of I are variables then sdepth $(I/J) \geq$ depth (I/J).

Stanley decompostions

A Stanley decomposition $\mathbb{D}$ of M is a direct sum of Z^n-graded K -vector spaces

$$\mathbb{D} : M = \bigoplus_{i=1}^{m} u_i K[Z_i]$$

where each $u_i \in M$ is homogeneous, $Z_i \subset X = \{x_1, ..., x_n\}$ and each $u_i K[Z_i]$ is a free $K[Z_i]$ - submodule of M.

We set sdepth $\mathbb{D} = \min\{|Z_i| : i = \overline{1, m}\}$, and

sdepth $M = \max\{$ sdepth $\mathbb{D} : \mathbb{D}$ is a Stanley decomposition of $M\}$ is called the Stanley depth of M.

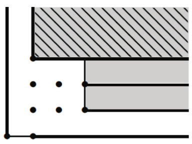
It is conjectured by Stanley [23] depth $M \leq$ sdepth M for all Z^n -graded S-modules M.

The Stanley depth for modules of the form I/J where $J \subset I \subset S = K[x_1, ..., x_n]$ are monomial ideals is a pure combinatorial invariant, in particular, it does not depend on the field K, while the depth is a homological invariant, and in the case of squarefree monomial ideal a topological invariant of the attached simplicial complex, which may very well depend on the field K.

An equivalent definition for the Stanley depth in our case can be given using partitions. Let $P_{\{I \setminus J\}}$ be the poset of all squarefree monomials of $I \setminus J$ with the order given by the divisibility. Let $\mathcal{P}$ be a partition of $P_{I \setminus J}$ in intervals $[u, v] = \{w \in P_{I \setminus J} : u|w, w|v\}$, let us say $P_{I \setminus J} = \cup_i [u_i, v_i]$, the union being disjoint. Define sdepth $\mathcal{P} = \min_i \deg v_i$ and the Stanley depth of I/J given by sdepth$_S$ $I/J = \max_{\mathcal{P}}$ sdepth $\mathcal{P}$, where $\mathcal{P}$ runs in the set of all partitions of $P_{\{I \setminus J\}}$ (see [5], [23]).

Example

The picture displays a Stanley decomposition of S/I and of I for the monomial ideal $I = (x_1x_2^3, x_1^3, x_2)$. The gray area represents the K-vector space spanned by the monomials in I. The hatched area, the fat lines and the isolated fat dots represent Stanley spaces of dimension 2,1, and 0, respectively.



$$I = x_1x_2^3K[x_1, x_2] \oplus x_1^3x_2^2K[x_1] \oplus x_1^3x_2K[x_1],$$

$$S/I = K[x_2] \oplus x_1K[x_1] \oplus x_1x_2K \oplus x_1x_2^2K \oplus x_1^2x_2K \oplus x_1^2x_2^2K.$$

Here we identify S/I with the K-subspace of S generated by all monomials $u \in S \setminus I$.

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