On the Hilbert function of a class of power ideals

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Abstract

In relation to the recent work [1] due to R. Fröberg, G. Ottaviani and B. Shapiro in connection with the Waring problem for polynomials, we are interested in study a particular class of *power ideals*.

In this poster, we will focus in particular on the computation of the Hilbert function of such ideals.

Basic notations

Fix a triple of positive integers (k, d, n) with $k \geq 2$. Let $S = \mathbb{C}[x_0, \ldots, x_n]$.

Definition. An homogeneous ideal $I \subset S$ is called a **power ideal** if I is generated by some powers $L_1^{d_1}, \ldots, L_s^{d_s}$ where the L_i's are linear forms.

Let $\mathbb{Z}_k$ be the group of cosets modulo k. We are interested in the following class of power ideals

$$I_{k,d,n} := \left(\phi_{\mathbf{g}}^D \mid \mathbf{g} \in \mathbb{Z}_k^{n+1}\right), \text{ with } D := (k-1)d,$$

where

$$\phi_{\mathbf{g}}(\mathbf{x}) := x_0 + \xi^{g_1} x_1 + \ldots + \xi^{g_n} x_n.$$

and ξ is a k^{th}-root of unity.

First properties

We'll denote the **weight** of $\mathbf{g} \in \mathbb{Z}_k^{n+1}$ as

$$\mathrm{wt}(\mathbf{g}) := \sum_{i=0}^n g_i.$$

Fact 1. $S_{i,\mathbf{g}} \neq 0 \Leftrightarrow i - \mathrm{wt}(\mathbf{g}) = mk$ for some $m \in \mathbb{N}$, in which case

$$\dim_{\mathbb{C}} S_{i,\mathbf{g}} = \binom{n+m}{m}.$$

We represent the generator $\phi_{\mathbf{0}}$ of our ideal $I_{k,d,n}$ as

$$\phi_{\mathbf{0}}^D(\mathbf{x}) = (x_0 + x_1 + \ldots + x_n)^D = \sum_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} \psi_{\mathbf{g}}(\mathbf{x}).$$

We have $\psi_{\mathbf{g}} \in S_{D,\mathbf{g}}$ and one can show that

$$\psi_{\mathbf{g}} \neq 0 \Leftrightarrow S_{D,\mathbf{g}} \neq 0.$$

Fact 2. $I_{k,d,n}$ is minimally generated by $\{\psi_{\mathbf{g}}\}$.

In order to compute the Hilbert function of $I_{k,n,d}$, we define the maps

$$\mu_{i,\mathbf{h}} : \bigoplus_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} S_{i,\mathbf{h}-\mathbf{g}} \xrightarrow{\cdot \psi_{\mathbf{g}}} S_{i+D,\mathbf{h}},$$

given by multiplications by $\psi_{\mathbf{g}}$.

Main result: the $k = 2$ case

From now on, we consider $\mathbf{k} = \mathbf{2}$, i.e. $D = d$ and the coefficients of the generators $\phi_{\mathbf{g}}$ are simply ± 1.

Lemma. (1) $\mu_{i,\mathbf{h}}$ is injective for $\mathrm{wt}(\mathbf{h}) \leq d - i$;
(2) $\mu_{i,\mathbf{h}}$ is surjective for $\mathrm{wt}(\mathbf{h}) \geq d - i$.

Denoting the quotient ring as $R = S/I_{2,d,n}$, we have $\mathrm{HF}(R; i) := \dim_{\mathbb{C}} R_i = \dim_{\mathbb{C}} S_i - \dim_{\mathbb{C}} I_i =$

$$= \sum_{\substack{\mathbf{h} \in \mathbb{Z}_k^{n+1} \\ \mathrm{wt}(\mathbf{h}) \leq d-i}} \dim_{\mathbb{C}} R_{i,\mathbf{h}} =: \sum_{\substack{\mathbf{h} \in \mathbb{Z}_k^{n+1} \\ \mathrm{wt}(\mathbf{h}) \leq d-i}} \mathrm{HF}(R; i, \mathbf{h}).$$

Theorem. (1) If $i < d$, then $\mathrm{HF}(R; i) = \binom{n+i}{n}$;
(2) if $i = j + d$ with $j = 0, \ldots, d$, then, for any $\mathbf{h} \in \mathbb{Z}_k^{n+1}$ s.t. $S_{i,\mathbf{h}} \neq 0$ and $\mathrm{wt}(\mathbf{h}) < d - j$,

$$\mathrm{HF}(R; i, \mathbf{h}) =$$
$$= \dim_{\mathbb{C}}(S_{i,\mathbf{h}}) - \sum_{\substack{j-k \in 2\mathbb{N} \\ k \geq 0}} \binom{n+1}{k} \dim_{\mathbb{C}}(S_{j-k,\mathbf{0}}).;$$

The multicyclic gradation

Consider the extension of the usual of the projection map,

$$\pi_{n+1} : \mathbb{N}^{n+1} \to \mathbb{Z}_k^{n+1}, \ (a_0, \ldots, a_n) \mapsto ([a_0]_k, \ldots, [a_n]_k).$$

For any monomial $\mathbf{x}^{\mathbf{a}} := x_0^{a_0} \ldots x_n^{a_n}$, we define the **multicyclic degree** as

$$\boxed{\mathrm{mcdeg}(\mathbf{x}^{\mathbf{a}}) := \pi_{n+1}(\mathbf{a}) \in \mathbb{Z}_k^{n+1},}$$

which yelds to a *multicyclic grading on S*. Combining with the total degree $S = \bigoplus_{i \in \mathbb{N}} S_i$, we get the *bigrading*

$$S = \bigoplus_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} S_{\mathbf{g}} = \bigoplus_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} \bigoplus_{i \in \mathbb{N}} S_{i,\mathbf{g}}, \text{ where } S_{i,\mathbf{g}} := S_i \cap S_{\mathbf{g}}.$$

Final comments

(1) The Theorem turns into an implementable algorithm which works very fast even for large numbers.
(2) From a geometrical point of view, the ideals $I_{k,d,n}$ corresponds to particular schemes of fat points.
(3) We believe that the Lemma can be generalized for $k \geq 2$ by replacing $(d-i)$ with $(k-1)(d-i)$.
A generalization of the Theorem would follow.

References

[1] Fröberg, Ralf and Ottaviani, Giorgio and Shapiro, Boris, *On the Waring problem for polynomial rings*, Proceedings of the National Academy of Sciences, vol.109.