Almost Vanishing Polynomials

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CoCoASchool 2013
Osnabrück

June 2013

Vanishing polynomials

Let

- K be a field (usually $K = \mathbb{R}$)
- $P = K[x_1, \ldots, x_n]$ polynomial ring (with $n \geq 1$)
- $\mathbb{X} = \{p_1, \ldots, p_s\} \subseteq K^n$ affine point set (distinct points)
- $\mathcal{I}(\mathbb{X}) = \{f \in P \mid f(p_i) = 0, \forall p_i \in \mathbb{X}\}$ is the **vanishing ideal** of $\mathbb{X}$
- A Gröbner basis of $\mathcal{I}(\mathbb{X})$ can be computed using the **Buchberger-Möller** algorithm

Almost "vanishing" polynomials

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- In the **approximate** case the constraint that f has to vanish **exactly** at the points is not always what we want...

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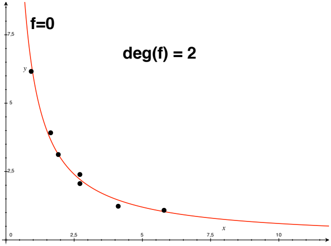
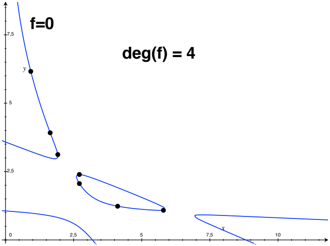
- **Example:** $\mathbb{X} = 10$ points of $\mathbb{R}^2$ close to $xy = k$



What we want in the **approximate** case



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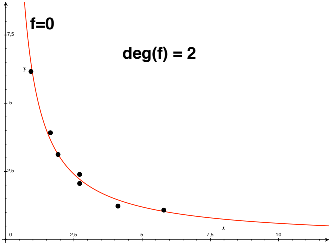

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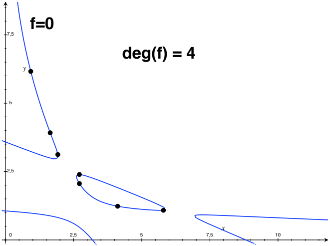

What we get in the **exact** case

- If we consider a single polynomial f the requirements are:
  - f is an **approximation** of $\mathbb{X}$ (in some sense)
  - f enjoys certain **features** (usually $\deg(f)$ small or upper bounded)

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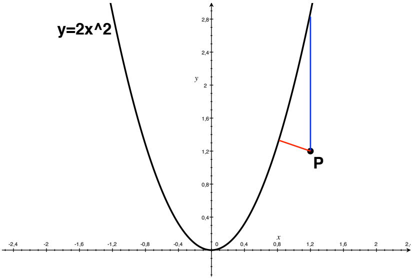

Analytic approach vs Geometrical approach II

- We observe that:
  - **(1)** **Analytic** $\not\Rightarrow$ **Geometrical**:
    $|f(p)|$ small $\not\Rightarrow$ $\mathrm{dist}(p, f = 0)$ small
  - **(2)** **Geometrical** $\not\Rightarrow$ **Analytic**
    $\mathrm{dist}(p, f = 0)$ small $\not\Rightarrow$ $|f(P)|$ small

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- **Remark:**
  it depends on the (local differential) geometry of the hypersurface $z = f(x_1, \dots, x_n)$ in $\mathbb{A}^{n+1}(\mathbb{R})$.

Example 1

- Let $f(x,y) = x^2 + \frac{1}{100}y^2 - \frac{1}{100} \in \mathbb{R}[x,y]$ e $p = (0,2)$.
- We have

$$|f(p)| = 0.03 \text{ “small”}$$

$$\mathrm{dist}(p, f=0) = 1 \text{ “big”}$$

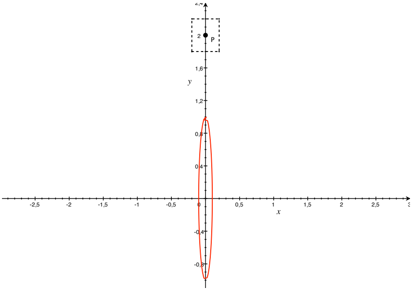


Figure : $x^2 + \frac{1}{100}y^2 - \frac{1}{100} = 0$ and point $p = (0,2)$

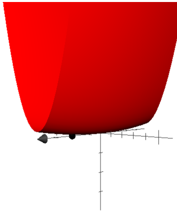


Figure : $z = x^2 + \frac{1}{100}y^2 - \frac{1}{100} = 0$ and point $p' = (0,2,0)$

Example 2

- Let $f(x,y) = y - 10x^2 \in \mathbb{R}[x,y]$ e $p = (1.1, 10)$.
- We have

$$|f(p)| = 2.1 \text{ “big”}$$
$$\mathrm{dist}(p, f = 0) \approx 0.1 \text{ “small”}$$

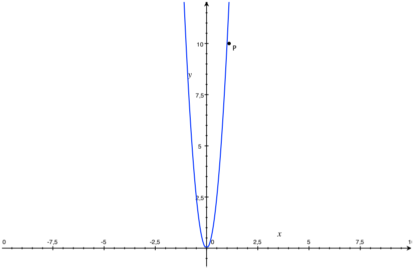


Figure : $y - 10x^2 = 0$ and point $p = (1.1, 10)$

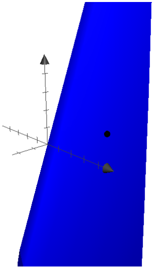


Figure : $z = y - 10x^2 = 0$ and point $p' = (1.1, 10, 0)$

Exact case - the BM algorithm

INPUT: $\mathbb{X}$ = set of s distinct points of $\mathbb{R}^n$, σ = term ordering
OUTPUT: $\mathcal{G}$ = Gröbner basis of $\mathcal{I}(\mathbb{X})$

S1 $\mathcal{O} = \{1\}$, $L = \{x_1, \dots, x_n\}$, $\mathcal{G} = \emptyset$.

S2 $L = \emptyset$? **YES**: return $\mathcal{G}$ and stop.
NO: $t = \min_\sigma(L)$, $L = L \setminus \{t\}$.

S3 Let $\rho(\mathbb{X})$ = residual of the LSP $M(\mathbb{X})\alpha(\mathbb{X}) = t(\mathbb{X})$
Is $t(\mathbb{X})$ linearly dependent on $M_{\mathcal{O}}(\mathbb{X})$?
YES $\Leftrightarrow \rho(\mathbb{X}) = \mathbf{0}$: continue with **S4**.
NO $\Leftrightarrow \rho(\mathbb{X}) \neq \mathbf{0}$: continue with **S5**.

S4 $f = t - \sum_i \alpha_i(\mathbb{X})t_i \to \mathcal{G}$; $L = L \setminus \{\text{multip. of } t\}$. Continue with **S2**.

S5 $t \to \mathcal{O}$; $z \to L$ where $z = x_i t$ and z not multip. of L or $\mathrm{LT}_\sigma\{\mathcal{G}\}$. Continue with **S2**.

Approximate case

- **Set of limited precision points:**
  $\mathbb{X} \subseteq \mathbb{R}^n$ set of s distinct **points** whose coords are known by less than **tolerance** ε.
- e_{kj} : perturbation in the j-th coordinate of the k-th point of $\mathbb{X}$.
- $\mathbf{e} = (e_{11}, \ldots, e_{s1}, \ldots, e_{1n}, \ldots, e_{sn})$
- $\mathbb{X}(\mathbf{e})$: any perturbation of $\mathbb{X}$.
- $\mathbb{X}(\mathbf{e})$: **admissible perturb.** of $\mathbb{X}$ w.r.t. $\varepsilon \Leftrightarrow \|\mathbf{e}\|_\infty \leq \varepsilon$.

Approximate case - the NBM algorithm

INPUT: $\mathbb{X}$ = set of s distinct points of $\mathbb{R}^n$, $\varepsilon \in \mathbb{R}^+$ = tolerance

OUTPUT: $f \in \mathbb{P}$ s.t.

- $d = \deg(f) \leq \min\{\deg(g) \mid g \in \mathcal{I}(\mathbb{X})\}$
- $\frac{\|f(\mathbb{X})\|_2}{\|f\|_2} < nsd\varepsilon$

Substitute **S2, S3** of BM Alg. with:

S2' $\mathcal{G} \neq \emptyset$? **YES**: return f and stop.

NO: $t = \min_\sigma(L)$, $L = L \setminus \{t\}$

S3' Let $\rho(\mathbb{X})$ = residual of the LSP: $M(\mathbb{X})\alpha(\mathbb{X}) = t(\mathbb{X})$

Is $t(\mathbb{X})$ num. lin. dep. on $M_{\mathcal{O}}(\mathbb{X})$? If

$$|\rho(\mathbb{X})| < \varepsilon |I - M_{\mathcal{O}}(\mathbb{X})M_{\mathcal{O}}^{+}(\mathbb{X})| \sum_{k=1}^{n} \left| \frac{\partial \rho}{\partial x_k}(\mathbb{X}) \right| \tag{1}$$

then **YES**: continue with **S4**.

otherwise **NO**: continue with **S5**.

Approximate case - the PassingClose algorithm

INPUT: $\mathbb{X}$ = set of s distinct points of $\mathbb{R}^n$, $\varepsilon \in \mathbb{R}^+$ = tolerance

OUTPUT: $f \in \mathbb{P}$ s.t.

- $d = \deg(f) \leq \min\{\deg(g) \mid g \in \mathcal{I}(\mathbb{X})\}$
- $\exists\, \mathbb{X}(\mathbf{e}^*)$ admiss. perturb. of $\mathbb{X}^\varepsilon$ s.t. $f(\mathbb{X}(\mathbf{e}^*)) = 0$

Substitute **S2, S3** of BM Alg. with:

S2' $\mathcal{G} \neq \emptyset$? **YES**: return f and stop.

NO: $t = \min_\sigma(L)$, $L = L \setminus \{t\}$

S3' Let $\rho(\mathbf{e})$ = residual of the LSP: $M(\mathbb{X}(\mathbf{e}))\alpha = t(\mathbb{X}(\mathbf{e}))$

If **condition (1)** is satisfied then compute $\mathbf{e}^*$ s.t.

$$\|\mathbf{e}^*\| = \min\{\|\mathbf{e}\| \mid \rho(\mathbf{e}) = 0\}$$

If $\|\mathbf{e}^*\| < \varepsilon$ then **YES**: continue with **S4**.

otherwise **NO**: continue with **S5**.

- In CoCoA5 there are (soon for any platform) the following algorithms:

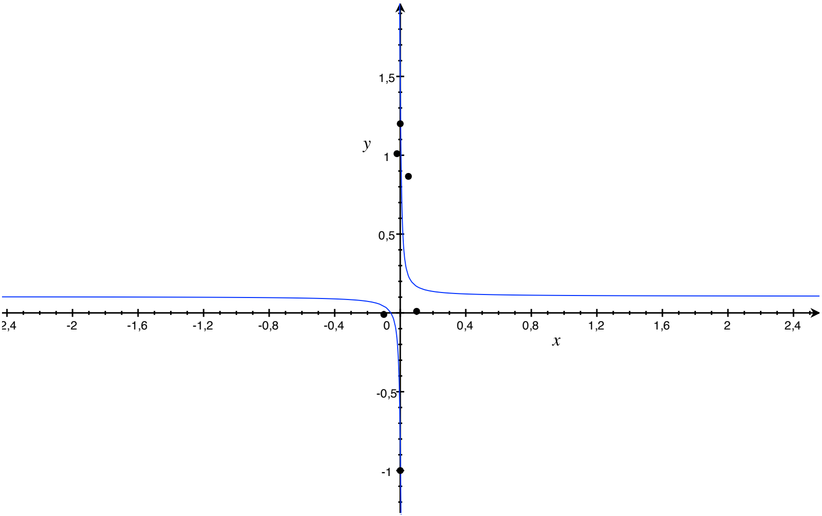
  TmpNBM and **ClosePassingPoly**

- Let's consider the following example

$$\begin{aligned}\mathbb{X} &= \{(-0.02, 1.01), (0, 1.2), (0.1, 0.01),\\ &\quad (-0.1, -0.01), (0, -1), (0.05, 0.866)\}\end{aligned}$$

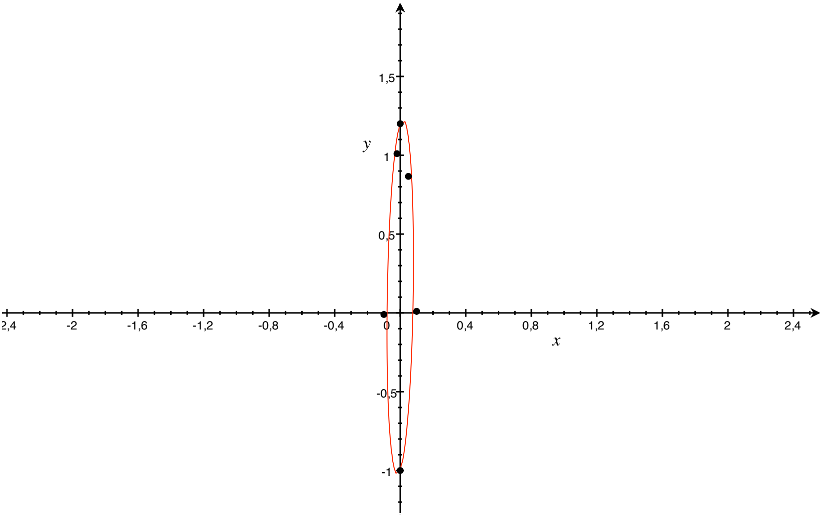
with $\varepsilon = (0.03, 0.03)$. We do the computation in CoCoA.....

Example



Output of NBM algorithm

Example



Output of PassingClose algorithm

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