Sets of Points and Mathematical Models

Lorenzo Robbiano

Università di Genova
Dipartimento di Matematica

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Abstract

1. Ideals of Points.
   1. Gröbner Bases, reduced Gröbner Bases.
   2. Homogenization.
   3. Hilbert Functions.
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   1. Border Bases.
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5. Approximate Interpolation on Finite Sets of Points (**Kreuzer**).

Motivation

Where do we encounter points? A few examples are

1. Algebraic geometry: linear sections of algebraic schemes
2. Interpolation
3. Design of experiments
4. Pixels and images

PART 1

Ideals of Points

Gröbner Bases

Theorem

Given a term ordering σ, *for a set of elements* $G = \{g_1, \dots, g_s\} \subseteq P^r \setminus \{0\}$ *which generates a submodule* $M = \langle g_1, \dots, g_s \rangle \subseteq P^r$, *let* $\xrightarrow{G}$ *be the rewrite rule defined by* G, *let* $\mathcal{G}$ *be the tuple* $(g_1, \dots, g_s)$.
Then the following conditions are equivalent.

$A_1)$ *For every* $m \in M \setminus \{0\}$ *, there are* $f_1, \dots, f_s \in P$ *such that* $m = \sum_{i=1}^s f_i g_i$ *and* $\mathrm{LT}_\sigma(m) \geq_\sigma \mathrm{LT}_\sigma(f_i g_i)$ *for all* $i = 1, \dots, s$ *such that* $f_i g_i \neq 0$.

$A_2)$ *For every element* $m \in M \setminus \{0\}$ *, there are* $f_1, \dots, f_s \in P$ *such that* $m = \sum_{i=1}^s f_i g_i$ *and* $\mathrm{LT}_\sigma(m) = \max_\sigma\{\mathrm{LT}_\sigma(f_i g_i) \mid i \in \{1, \dots, s\}, f_i g_i \neq 0\}$.

$B_1)$ *The set* $\{\mathrm{LT}_\sigma(g_1), \dots, \mathrm{LT}_\sigma(g_s)\}$ *generates the* $\mathbb{T}^n$ *-monomodule* $\mathrm{LT}_\sigma\{M\}$.

$B_2)$ *The set* $\{\mathrm{LT}_\sigma(g_1), \dots, \mathrm{LT}_\sigma(g_s)\}$ *generates the* P *-submodule* $\mathrm{LT}_\sigma(M)$ *of* P^r .

$C_1)$ *For an element* $m \in P^r$ *, we have* $m \xrightarrow{G} 0$ *if and only if* $m \in M$.

$C_2)$ *For every* $m_1 \in P^r$ *, there is a unique* $m_2 \in P^r$ *such that* $m_1 \xrightarrow{G} m_2$ *and* m_2 *is irreducible w.r.t* $\xrightarrow{G}$.

$C_3)$ *If* $m_1, m_2, m_3 \in P^r$ *satisfy* $m_1 \xrightarrow{G} m_2$ *and* $m_1 \xrightarrow{G} m_3$ *, then there exists an element* $m_4 \in P^r$ *such that* $m_2 \xrightarrow{G} m_4$ *and* $m_3 \xrightarrow{G} m_4$.

$D_1)$ *Every homogeneous element of* $\mathrm{Syz}(\mathrm{LM}_\sigma(\mathcal{G}))$ *has a lifting in* $\mathrm{Syz}(\mathcal{G})$.

$D_2)$ *There exists a finite homogeneous system of generators of* $\mathrm{Syz}(\mathrm{LM}_\sigma(\mathcal{G}))$ *which have a lifting in* $\mathrm{Syz}(\mathcal{G})$.

Buchberger's Algorithm

Let $f_1, \dots, f_s$ be non-zero elements in P and let I be the ideal of P generated by $\{f_1, \dots, f_s\}$.

1. **(Initialization)**
   Pairs $= \emptyset$, *the pairs*; Gens $= (f_1, \dots, f_s)$, *the generators of* I; $G = \emptyset$, *the* σ*-Gröbner basis of* I *under construction.*

2. **(Main loop)**
   While Gens $\neq \emptyset$ or Pairs $\neq \emptyset$ do
   - (2a) **choose** $f \in$ Gens and remove it from Gens, or a pair $(f_i, f_j) \in$ Pairs, remove it from Pairs, and let $f = \mathrm{S}(f_i, f_j)$;
   - (2b) compute **a remainder** $g := \mathrm{Rem}(f, G)$;
   - (2c) if $g \neq 0$ add g to G and the pairs $\{(g, f_i) \mid f_i \in G\}$ to Pairs.

3. **(Output)**
   Return G.

This is an algorithm which returns **a σ-Gröbner basis of** I, *whatever choices are made in step (2a)* and *whatever remainder is computed in step (2b).*

Reduced Gröbner Bases

Definition

Let $G = \{g_1, \dots, g_s\} \subseteq P^r \setminus \{0\}$ and $M = \langle g_1, \dots, g_s \rangle$. We say that G is a reduced σ -Gröbner basis of M if the following conditions are satisfied.

- For $i = 1, \dots, s$, we have $\mathsf{LC}_\sigma(g_i) = 1$.
- The set $\{\mathsf{LT}_\sigma(g_1), \dots, \mathsf{LT}_\sigma(g_s)\}$ is a minimal system of generators of $\mathsf{LT}_\sigma(M)$.
- For $i = 1, \dots, s$, we have $\mathrm{Supp}(g_i - \mathsf{LT}_\sigma(g_i)) \cap \mathsf{LT}_\sigma\{M\} = \emptyset$.

Theorem

(Existence and Uniqueness of Reduced Gröbner Bases)
For every P -submodule $M \subseteq P^r$, there exists a unique reduced σ -Gröbner basis.

Homogenization

The algebraic process of homogenization corresponds to the geometric process of taking the (weighted) projective closure of an affine scheme.

Proposition

Let the polynomial ring $P = K[x_1, \dots, x_n]$ *be graded by a row of positive integers* $W = (w_1 \ \cdots \ w_n)$. *Given an ideal* I *in* P*, consider the following sequence of instructions.*

1. *Choose a non-singular matrix* $V \in \mathrm{Mat}_n(\mathbb{Z})$ *of the form* $V = \binom{W}{W'}$ *, where* $W' \in \mathrm{Mat}_{n-1,n}(\mathbb{Z})$.
2. *Compute a Gröbner basis* $\{g_1, \dots, g_s\}$ *of* I *with respect to* $\mathtt{Ord}(V)$.
3. *Return the ideal* $(g_1^{\mathrm{hom}}, \dots, g_s^{\mathrm{hom}})$ *and stop.*

This is an algorithm which computes $I^{\mathrm{hom}} = (g_1^{\mathrm{hom}}, \dots, g_s^{\mathrm{hom}})$.

Moreover, the homogenizing indeterminate is a non zero-divisor modulo I^{hom} .

Standard Hilbert Functions and Hilbert Series

Let $M = \oplus_{d\in\mathbb{Z}} M_d$ be a finitely generated standard graded module over P.
The Hilbert Function of M is the function

$$\mathsf{HF}_M : \mathbb{Z} \longrightarrow \mathbb{Z} \quad \text{defined by} \quad i \longrightarrow \dim_K(M_i)$$

The Hilbert Series of M is the Laurent series

$$\mathsf{HS}_M(z) = \sum_{i\geq\alpha} \mathsf{HF}_M(i) z^i$$

Theorem

Let σ be a module term ordering. Then we have

$$\mathsf{HS}_M(z) = \mathsf{HS}_{\mathsf{LT}_\sigma(M)}(z)$$

Tools for computing Hilbert Series

Theorem

Let $f \in P$ be a homogeneous polynomial of degree d. Then we have the following facts.

1. *There exists an exact sequence of graded P-modules*

$$0 \longrightarrow [M/\langle 0 :_M (f)\rangle](-d) \xrightarrow{f} M \longrightarrow M/fM \longrightarrow 0$$

2. *f is a non-zerodivisor for M if and only if $\mathsf{HS}_{M/fM}(z) = (1-z^d)\,\mathsf{HS}_M(z)$.*

Theorem

Let M be a non-zero finitely generated standard graded P-module, and let $\alpha(M) = \min\{i \in \mathbb{Z} \mid M_i \neq 0\}$. Then the Hilbert series of M has the form

$$\mathsf{HS}_M(z) = \frac{z^{\alpha(M)}\,\mathsf{HN}_M(z)}{(1-z)^n}$$

where $\mathsf{HN}_M(z) \in \mathbb{Z}[z]$ and $\mathsf{HN}_M(0) = \mathsf{HF}_M(\alpha(M)) > 0$.
In particular we have $\mathsf{HS}_P(z) = \frac{1}{(1-z)^n}$.

Affine Hilbert Series

Assumption

1. *By* $\langle P_{\leq i}\rangle$ *we shall denote the* K *-vector space of all polynomials of degree* $\leq i$ *, including the zero polynomial.*
2. *The* K *-vector space* $\langle I_{\leq i}\rangle$ *is the vector subspace of* $\langle P_{\leq i}\rangle$ *which consists of the polynomials of degree* $\leq i$ *in* I *.*
3. *Since* $\langle I_{\leq i}\rangle = \langle P_{\leq i}\rangle \cap I$ *, we can view the vector space* $\langle P_{\leq i}\rangle / \langle I_{\leq i}\rangle$ *as a vector subspace of* P/I *.*

Definition

1. The map $\mathsf{HF}^a_{P/I} : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $\mathsf{HF}^a_{P/I}(i) = \dim_K(\langle P_{\leq i}\rangle / \langle I_{\leq i}\rangle)$ for $i \in \mathbb{Z}$ is called the affine Hilbert function of P/I .
2. The power series $\mathsf{HS}^a_{P/I}(z) = \sum_{i\geq 0} \mathsf{HF}^a_{P/I}(i)\, z^i \in \mathbb{Z}[[z]]$ is called the affine Hilbert series of P/I .

Affine Hilbert Series: Example and Properties

Proposition

(Basic Properties of Affine Hilbert Functions)

Let σ *be a* degree compatible *term ordering on* $\mathbb{T}^n$*, and let* $W = (1 \ 1 \ \cdots \ 1)$ *be the matrix defining the standard grading on* P *.*

- *For every* $i \in \mathbb{Z}$ *, we have* $\mathrm{HF}^a_{P/I}(i) = \sum_{j=0}^{i} \mathrm{HF}_{P/\,\mathrm{LT}_\sigma(I)}(j)$ *. In particular, we have* $\mathrm{HF}^a_{P/I}(i) = \mathrm{HF}^a_{P/\,\mathrm{LT}_\sigma(I)}(i)$ *for all* $i \in \mathbb{Z}$ *.*
- *For every* $i \in \mathbb{Z}$ *, we have* $\mathrm{HF}^a_{P/I}(i) = \mathrm{HF}^a_{P/\,\mathrm{DF}_W(I)}(i)$ *.*
- *Let* x_0 *be a homogenizing indeterminate, and let* $\overline{P} = K[x_0, \ldots, x_n]$ *be standard graded. Then we have* $\mathrm{HF}^a_{P/I}(i) = \mathrm{HF}_{\overline{P}/I^{\mathrm{hom}}}(i)$ *for all* $i \in \mathbb{Z}$ *.*

Example

- Consider the affine K-algebra $R = K[x]/(x^3)$.
- We have $\mathrm{HF}^a_R(i) = \min\{i+1, 3\}$ for $i \geq 0$, 0 otherwise.
- It is easy to see that R is isomorphic to $R' = K[x,y]/(xy, x^2 - y)$.
- In this case we calculate $\mathrm{HF}^a_{R'}(i) = 3$ for $i \geq 1$, $\mathrm{HF}^a_{R'}(0) = 1$.
- These two affine Hilbert functions differ, because they differ for $i = 1$.

Affine Hilbert Series: Computation

Proposition

Let σ *be a degree compatible term ordering on* $\mathbb{T}^n$ *, let* x_0 *be a homogenizing indeterminate, and let* $\overline{P} = K[x_0, \ldots, x_n]$ *.*

- *We have* $\mathrm{HS}^a_{P/I}(z) = \frac{\mathrm{HS}_{P/\,\mathrm{LT}_\sigma(I)}(z)}{(1-z)}$
- *We have* $\mathrm{HS}^a_{P/I}(z) = \mathrm{HS}_{\overline{P}/I^{\mathrm{hom}}}(z)$.

Hilbert Polynomial

The last important information about Hilbert functions is the following. Assume that the grading is standard.

1. The Hilbert function of a finitely generated graded module is an integer function of polynomial type.
2. The integer valued polynomial associated to HF_M is called the Hilbert polynomial and denoted by $\mathsf{HP}_M(t)$. Hence $\mathsf{HF}_M(i) = \mathsf{HP}_M(i)$ for large i .

Consequently, if I and J are two homogeneous ideals in P with the same saturation, then $\mathsf{HP}_{P/I} = \mathsf{HP}_{P/J}$.

Zero-dimensional schemes

Theorem

(Finiteness Criterion)

Let σ be a term ordering on $\mathbb{T}^n$. Let $\mathcal{S}$ be a system of polynomial equations, and let I be the corresponding ideal. The following conditions are equivalent.

- *The system of equations $\mathcal{S}$ has only finitely many solutions.*
- *For $i = 1, \ldots, n$, we have $I \cap K[x_i] \neq (0)$.*
- *The K-vector space P/I is finite-dimensional.*
- *The set $\mathbb{T}^n \setminus \mathsf{LT}_\sigma\{I\}$ is finite.*
- *For every $i \in \{1, \ldots, n\}$, there exists a number $\alpha_i \geq 0$ such that we have $x_i^{\alpha_i} \in \mathsf{LT}_\sigma(I)$.*

Ideals of affine and projective Points.

In Computational Commutative Algebra 2 we wrote:

When one starts to reduce deep problems in algebraic geometry to their essential parts, it frequently turns out that at their core lies a question which has been studied for a long time, and sometimes this question is related to finite sets of points.

Definition

Let K be a field and $P = K[x_1, \dots, x_n]$.

- An element $p = (c_1, \dots, c_n)$ of K^n is also called a K -rational point. The numbers $c_1, \dots, c_n \in K$ are called the coordinates of p .
- A finite set $\mathbb{X} = \{p_1, \dots, p_s\}$ of distinct K -rational points $p_1, \dots, p_s \in K^n$ is called an affine point set.
- The vanishing ideal $\mathcal{I}(\mathbb{X}) \subseteq P$ of an affine point set $\mathbb{X} \subseteq K^n$ is called an ideal of points.
- The K -algebra $P/\mathcal{I}(\mathbb{X})$ is called the (affine) coordinate ring of $\mathbb{X}$.

First Properties

Example

Let $p = (c_1, \dots, c_n) \in K^n$ be a K-rational point and $\mathbb{X} = \{p\}$. The vanishing ideal of $\mathbb{X}$ is given by the ideal $\mathcal{I}(\mathbb{X}) = (x_1 - c_1, \dots, x_n - c_n) \subseteq P$.

In the following, we let $p_i = (c_{i1}, \dots, c_{in}) \in K^n$ with $c_{ij} \in K$ for $i = 1, \dots, s$ and $j = 1, \dots, n$, and we let $\mathbb{X}$ be the affine point set $\mathbb{X} = \{p_1, \dots, p_s\}$.

Proposition

(Basic Properties of Ideals of Points)

Let $\mathbb{X} = \{p_1, \dots, p_s\}$ be an affine point set as above.

1. *We have $\mathcal{I}(\mathbb{X}) = \mathcal{I}(p_1) \cap \cdots \cap \mathcal{I}(p_s)$.*
2. *The map $\varphi : P/\mathcal{I}(\mathbb{X}) \longrightarrow K^s$ defined by $\varphi(f + \mathcal{I}(\mathbb{X})) = (f(p_1), \dots, f(p_s))$ is an isomorphism of K-algebras. In particular, the ideal $\mathcal{I}(\mathbb{X})$ is zero-dimensional.*
3. *For any term ordering σ on $\mathbb{T}^n$, the set $\mathbb{T}^n \setminus \mathrm{LT}_\sigma\{I(\mathbb{X})\}$ consists of precisely s terms.*

Separators and Interpolators

Definition

Let $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq K^n$ be an affine point set, and let $\mathcal{X}$ be the tuple $(p_1, \dots, p_s)$.

- Let $i \in \{1, \dots, s\}$. A polynomial $f \in P$ is called a separator of p_i from $\mathbb{X} \setminus p_i$ if $f(p_i) = 1$ and $f(p_j) = 0$ for $j \neq i$.
- Let $a_1, \dots, a_s \in K$. A polynomial $f \in P$ is called an interpolator for the tuple $(a_1, \dots, a_s)$ at $\mathcal{X}$ if $f(p_i) = a_i$ for $i = 1, \dots, s$.

Proposition

Let $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq K^n$ be an affine point set, and let $\mathcal{X} = (p_1, \dots, p_s)$.

1. *For every $i \in \{1, \dots, s\}$, there exists a separator of p_i from $\mathbb{X} \setminus p_i$.*
2. *For all $(a_1, \dots, a_s) \in K^s$, there exists an interpolator for $(a_1, \dots, a_s)$ at $\mathcal{X}$.*

Lifting

Let $P = K[x_1, \dots, x_n]$ be positively graded by a matrix $W \in \mathrm{Mat}_{1,n}(\mathbb{Z})$, and let $\overline{P} = K[x_0, \dots, x_n]$ be graded by $\overline{W} = (1 \mid W)$.
Given a polynomial $F \in \overline{P}$, we let $F^{\mathrm{inf}} = F(0, x_1, \dots, x_n)$, and for an ideal $J \subseteq \overline{P}$, we let $J^{\mathrm{inf}} = (F^{\mathrm{inf}} \mid F \in J)$. Suppose that P and $\overline{P}$ are standard graded.
The map $\imath_0 : \mathbb{A}^n \longrightarrow \mathbb{P}^n$ defined by $\imath_0(p_1, \dots, p_n) = (1 : p_1 : \cdots : p_n)$ is injective.

- A homogeneous ideal J in $\overline{P}$ defines a zero-set $V = \mathcal{Z}^+(J)$ in $\mathbb{P}^n$.
- Its dehomogenization $J^{\mathrm{deh}} \subseteq P$ defines the affine part $V \cap \imath_0(\mathbb{A}^n)$ of V
- The homogeneous ideal $J^{\mathrm{inf}} \subseteq P$ defines $V \cap H^{\mathrm{inf}} = V \cap \mathcal{Z}^+(x_0)$, the set of points at infinity of V .

Definition

Let $I \subset P$ be a homogeneous ideal. A homogeneous ideal $J \subset \overline{P}$ is called a lifting of I with respect x_0 if the following conditions are satisfied:

- The indeterminate x_0 is a non-zero divisor for $\overline{P}/J$.
- We have $I = J^{\mathrm{inf}}$.

Example

The ideal I^{hom} is an x_0 -lifting of $\mathrm{DF}_W(I)$.

Distractions

Let K be an infinite field and choose n sequences $\pi_1, \ldots, \pi_n$ of elements of K in such a way that each sequence consists of pairwise distinct elements.
Thus we let $\pi_i = (c_{i1}, c_{i2}, \ldots)$ with $c_{ij} \in K$ and $c_{ij} \neq c_{ik}$ for $j \neq k$.

Definition

Let $\pi = (\pi_1, \ldots, \pi_n)$.

- For every term $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{T}^n$, the polynomial

$$D_\pi(t) = \prod_{i=1}^{\alpha_1}(x_1 - c_{1i}) \cdot \prod_{i=1}^{\alpha_2}(x_2 - c_{2i}) \cdots \prod_{i=1}^{\alpha_n}(x_n - c_{ni})$$

  is called the distraction of t with respect to π .
- Let I be a monomial ideal in P , and let $\{t_1, \ldots, t_s\}$ be the unique minimal monomial system of generators of I . Then we say that the ideal $D_\pi(I) = (D_\pi(t_1), \ldots, D_\pi(t_s))$ is the distraction of I with respect to π .

Distractions and Liftings

Theorem

(Liftings of Monomial Ideals)

Let $\pi = (\pi_1, \dots, \pi_n)$ *, let* $I \subset P$ *be a monomial ideal, and let* $\{t_1, \dots, t_s\}$ *be its minimal monomial system of generators.*

- *The distraction* $D_\pi(I)$ *is an intersection of finitely many ideals which are generated by linear polynomials.*
- *The distraction* $D_\pi(I)$ *is a radical ideal.*
- *For every term ordering* σ *on* $\mathbb{T}^n$ *, the set* $\{D_\pi(t_1), \dots, D_\pi(t_s)\}$ *is the reduced* σ *-Gröbner basis of* $D_\pi(I)$ *.*
- *We have* $D_\pi(I)^{\rm hom} = (D_\pi(t_1)^{\rm hom}, \dots, D_\pi(t_s)^{\rm hom})$ *in* $\overline{P}$ *. This ideal is an* x_0 *-lifting of* I *and a radical ideal.*
- *If* I *contains a pure power of each indeterminate, then* $D_\pi(I)$ *is the vanishing ideal of an affine point set, and* $D_\pi(I)^{\rm hom}$ *is the vanishing ideal of the same set viewed inside the projective space.*

More properties of Hilbert Functions

Proposition

Let $n, i \in \mathbb{N}_+$ *. The number* n *has a unique representation* $n = \binom{n(i)}{i} + \binom{n(i-1)}{i-1} + \cdots + \binom{n(j)}{j}$ *such that* $1 \le j \le i$ *and such that* $n(i), \ldots, n(j) \in \mathbb{N}$ *are natural numbers which satisfy* $n(i) > n(i-1) > \cdots > n(j) \ge j$ *.*

Definition

Let $n, i \in \mathbb{N}_+$ and let $n_{[i]} = \binom{n(i)}{i} + \cdots + \binom{n(j)}{j}$ of n in base i .
We denote the number $\binom{n(i)+1}{i+1} + \cdots + \binom{n(j)+1}{j+1}$ by $(n_{[i]})^+_+$.

Theorem

Macaulay growth theorem for ideals
Let K *be a field, let* $P = K[x_1, \ldots, x_n]$ *be standard graded, let* $I \subseteq P$ *be a homogeneous ideal, and let* $d \in \mathbb{N}_+$ *. Then we have*

$$\mathsf{HF}_{P/I}(d+1) \le ((\mathsf{HF}_{P/I}(d))_{[d]})^+_+$$

Here equality holds if I_d *is a* `Lex`*-segment space which satisfies* $I_{d+1} = P_1 \cdot I_d$ *.*

O-sequences and Castelnuovo functions

Definition

A function $H : \mathbb{Z} \longrightarrow \mathbb{Z}$ is called an **O-sequence** if it has the following properties.

- For $i < 0$, we have $H(i) = 0$, and $H(0) = 1$.
- There exists a number $r \in \mathbb{N}$ such that $H(i) = 0$ for $i \geq r$ and $H(i) \neq 0$ for $0 \leq i < r$.
- For $i = 1, \ldots, r-1$, we have $H(i+1) \leq (H(i)_{[i]})^+_+$.

Definition

Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a projective point set with homogeneous coordinate ring $R = \overline{P}/\mathcal{I}^+(\mathbb{X})$. Then the Hilbert function $\mathrm{HF}_R : \mathbb{Z} \longrightarrow \mathbb{Z}$ of R is also called the Hilbert function of $\mathbb{X}$ and denoted by $\mathrm{HF}_\mathbb{X}$.
Its first difference function $\Delta\,\mathrm{HF}_\mathbb{X} : \mathbb{Z} \longrightarrow \mathbb{Z}$ is called the Castelnuovo function of $\mathbb{X}$.

Hilbert Functions of finite Sets of Points

Proposition

Let $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}^n_K$ *be a projective point set.*

- *For* $i < 0$ *, we have* $\mathsf{HF}_{\mathbb{X}}(i) = 0$ *, and we have* $\mathsf{HF}_{\mathbb{X}}(0) = 1$ *.*
- *Let* $r_{\mathbb{X}} = \mathsf{ri}(\mathsf{HF}_{\mathbb{X}})$ *. Then we have* $\mathsf{HF}_{\mathbb{X}}(i) = s$ *for all* $i \geq r_{\mathbb{X}}$ *.*
- *We have* $\mathsf{HF}_{\mathbb{X}}(0) < \mathsf{HF}_{\mathbb{X}}(1) < \cdots < \mathsf{HF}_{\mathbb{X}}(r_{\mathbb{X}})$ *.*

Let $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{A}^n_K$ *be an affine point set. The same conclusion holds for the affine Hilbert function.*

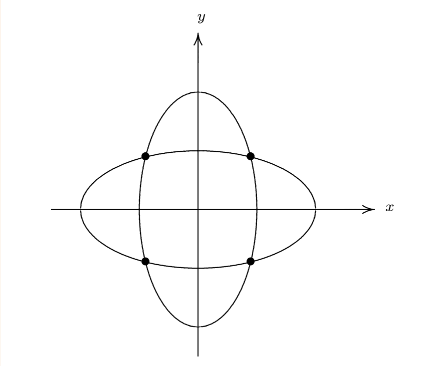
Corollary

The Castelnuovo function $\Delta H_{\mathbb{X}}$ *of a projective point set* $\mathbb{X}$ *is an O-sequence.*

PART 2

Border Bases and Border Basis Schemes

Two conics I



Example

Consider the polynomial system

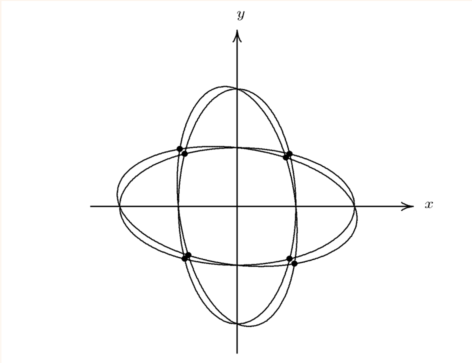
$$\begin{array}{rcccl} f_1 & = & \frac{1}{4}x^2 + y^2 - 1 & = & 0 \\ f_2 & = & x^2 + \frac{1}{4}y^2 - 1 & = & 0 \end{array}$$

$\mathbb{X} = \mathcal{Z}(f_1) \cap \mathcal{Z}(f_2)$ consists of the four points $\mathbb{X} = \{(\pm\sqrt{4/5}, \pm\sqrt{4/5})\}$.

The set $\{x^2 - \frac{4}{5},\, y^2 - \frac{4}{5}\}$ is the reduced Gröbner basis of the ideal $I = (f_1, f_2) \subseteq \mathbb{C}[x, y]$ with respect to $\sigma = \texttt{DegRevLex}$.

$\mathsf{LT}_\sigma(I) = (x^2,\, y^2)$, and the residue classes of the terms in $\mathbb{T}^2 \setminus \mathsf{LT}_\sigma\{I\} = \{1,\, x,\, y,\, xy\}$ form a $\mathbb{C}$ -vector space basis of $\mathbb{C}[x, y]/I$.

Two conics II



Now consider the slightly perturbed polynomial system

$$\begin{array}{rcccl} \tilde{f}_1 & = & \frac{1}{4}x^2 + y^2 + \varepsilon\, xy - 1 & = & 0 \\ \tilde{f}_2 & = & x^2 + \frac{1}{4}y^2 + \varepsilon\, xy - 1 & = & 0 \end{array}$$

The intersection of $\mathcal{Z}(\tilde{f}_1)$ and $\mathcal{Z}(\tilde{f}_2)$ consists of four perturbed points $\widetilde{\mathbb{X}}$ close to those in $\mathbb{X}$.

- The ideal $\tilde{I} = (\tilde{f}_1, \tilde{f}_2)$ has the reduced σ -Gröbner basis

$$\{x^2 - y^2,\ xy + \tfrac{5}{4\varepsilon}\, y^2 - \tfrac{1}{\varepsilon},\ y^3 - \tfrac{16\varepsilon}{16\varepsilon^2 - 25}\, x + \tfrac{20}{16\varepsilon^2 - 25}\, y\}$$

- Moreover, we have $\mathsf{LT}_\sigma(\tilde{I}) = (x^2,\, xy,\, y^3)$ and $\mathbb{T}^2 \setminus \mathsf{LT}_\sigma\{\tilde{I}\} = \{1,\, x,\, y,\, y^2\}$.

On Border Bases

Definition

Let $\mathcal{O}$ be a non-empty subset of $\mathbb{T}^n$.

- The closure of $\mathcal{O}$ is the set $\overline{\mathcal{O}}$ of all terms in $\mathbb{T}^n$ which divide one of the terms of $\mathcal{O}$.
- The set $\mathcal{O}$ is called order ideal or factor closed if $\overline{\mathcal{O}} = \mathcal{O}$, i.e. $\mathcal{O}$ is closed under forming divisors.

Definition

Let $\mathcal{O} \subseteq \mathbb{T}^n$ be an order ideal.
The border of $\mathcal{O}$ is the set $\partial\mathcal{O} = \mathbb{T}^n \cdot \mathcal{O} \setminus \mathcal{O} = (x_1\mathcal{O} \cup \cdots \cup x_n\mathcal{O}) \setminus \mathcal{O}$.
The first border closure of $\mathcal{O}$ is the set $\overline{\partial\mathcal{O}} = \mathcal{O} \cup \partial\mathcal{O}$.

It is possible to construct a Border Division Algorithm.

Border Bases

The basic idea of border basis theory is to describe a zero-dimensional ring P/I by an order ideal of monomials $\mathcal{O}$ whose residue classes form a K-basis of P/I and by the multiplication matrices of this basis.

Let K be a field, let $P = K[x_1, \dots, x_n]$, and let $\mathbb{T}^n$ be the monoid of terms.

Definition (Border Prebases)

Let $\mathcal{O}$ have μ elements and $\partial\mathcal{O}$ have ν elements.
A set of polynomials $G = \{g_1, \dots, g_\nu\}$ in P is called an $\mathcal{O}$-border prebasis if the polynomials have the form $g_j = b_j - \sum_{i=1}^{\mu} \alpha_{ij} t_i$ with $\alpha_{ij} \in K$ for $1 \leq i \leq \mu$, $1 \leq j \leq \nu$, $b_j \in \partial\mathcal{O}$, $t_i \in \mathcal{O}$.

Definition (Border Bases)

Let $G = \{g_1, \dots, g_\nu\}$ be an $\mathcal{O}$-border prebasis, and let $I \subseteq P$ be an ideal containing G. The set G is called an $\mathcal{O}$-border basis of I if the residue classes $\overline{\mathcal{O}} = \{\bar{t}_1, \dots, \bar{t}_\mu\}$ form a K-vector space basis of P/I.

If the zero-dimensional scheme represented by P/I is a tuple of distinct points $\mathbb{X} = (p_1, \dots, p_s)$, then a tuple of s polynomials $(f_1, \dots, f_s)$ is a basis modulo I if and only if the evaluation matrix $(f_j(p_i))$ is invertible.

Two conics III

What are the border bases in the two cases of the conics and the perturbed conics?

Two conics

$$\{x^2 - \tfrac{4}{5}, \qquad x^2y - \tfrac{4}{5}y,$$
$$xy^2 - \tfrac{4}{5}x, \qquad y^2 - \tfrac{4}{5}\}$$

Two perturbed conics

$$\{x^2 + \tfrac{4}{5}\,\varepsilon xy - \tfrac{4}{5}, \qquad x^2y - \tfrac{16\varepsilon}{16\varepsilon^2-25}\,x + \tfrac{20}{16\varepsilon^2-25}\,y,$$
$$xy^2 + \tfrac{20}{16\varepsilon^2-25}\,x + \tfrac{16\varepsilon}{16\varepsilon^2-25}\,y, \qquad y^2 + \tfrac{4}{5}\,\varepsilon xy - \tfrac{4}{5}\}$$

Existence and Uniqueness of Border Bases

Proposition

Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal, let $I \subseteq P$ be a zero-dimensional ideal, and assume that the residue classes of the elements of $\mathcal{O}$ form a K-vector space basis of P/I.

- *There exists a unique $\mathcal{O}$-border basis of I.*
- *Let G be an $\mathcal{O}$-border prebasis whose elements are in I. Then G is the $\mathcal{O}$-border basis of I.*
- *Let k be the field of definition of I. Then the $\mathcal{O}$-border basis of I is contained in $k[x_1, \ldots, x_n]$.*

Proposition

Let σ be a term ordering on $\mathbb{T}^n$, and let $\mathcal{O}_\sigma(I)$ be the order ideal $\mathbb{T}^n \setminus \mathsf{LT}_\sigma\{I\}$. Then there exists a unique $\mathcal{O}_\sigma(I)$-border basis G of I, and the reduced σ-Gröbner basis of I is the subset of G corresponding to the corners of $\mathcal{O}_\sigma(I)$.

Commuting matrices

The following is a fundamental fact.

B. Mourrain: *A new criterion for normal form algorithms*, AAECC Lecture Notes in Computer Science **1719** (1999), 430–443.

Theorem (Border Bases and Commuting Matrices)

Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ *be an order ideal, let* $G = \{g_1, \ldots, g_\nu\}$ *be an* $\mathcal{O}$ *-border prebasis, and let* $I = (g_1, \ldots, g_\nu)$ *. Then the following conditions are equivalent.*

1. *The set* G *is an* $\mathcal{O}$ *-border basis of* I *.*
2. *The multiplication matrices of* G *are pairwise commuting.*

In that case the multiplication matrices represent the multiplication endomorphisms of P/I *with respect to the basis* $\{\bar{t}_1, \ldots, \bar{t}_\mu\}$ *.*

A glimpse at punctual Hilbert schemes

- Punctual Hilbert schemes are schemes which parametrize all the zero-dimensional projective subschemes of $\mathbb{P}^n$ which share the same multiplicity.
- Every zero-dimensional sub-scheme of $\mathbb{P}^n$ is contained in a standard open set which is an affine space, say $\mathbb{A}^n \subset \mathbb{P}^n$.
- There is a one-to-one correspondence between zero-dimensional ideals in $P = K[x_1, \ldots, x_n]$ and zero-dimensional saturated homogeneous ideals in $\overline{P} = K[x_0, x_1, \ldots, x_n]$. The correspondence is set via homogenization and dehomogenization.

Zero-dimensional ideals and monomial bases

- We know that the affine Hilbert function of P/I is the same as the Hilbert function of $\overline{P}/I^{\mathrm{hom}}$, and hence also the Hilbert polynomial and the difference function are equal.
- The disadvantage of covering $\mathbb{P}^n$ with affine spaces is compensated by the fact that $\dim_K(P/I) < \infty$, while $\dim_K(\overline{P}/I^{\mathrm{hom}}) = \infty$.
- Therefore, in our setting we can (and we do) consider finite bases of the quotient ring P/I viewed as a K-vector space.
- We restrict ourselves to monomial bases which are order ideals.
- Their complement is a monomial ideal.

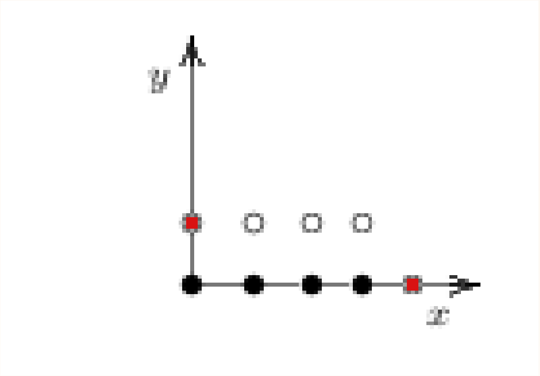
An Example: Hilbert Polynomial = 4

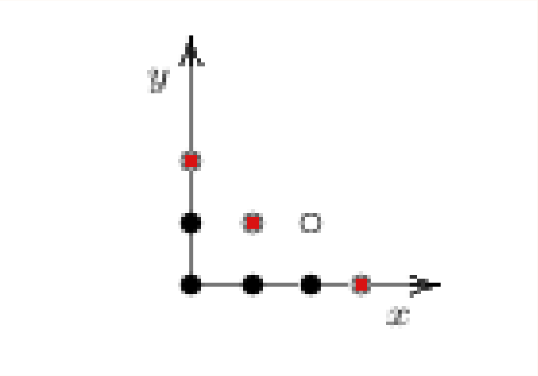
- Zero-dimensional subschemes of $\mathbb{P}^2$ with Hilbert polynomial 4 correspond to saturated homogeneous ideals I such that if P denotes the polynomial ring $K[x, y, z]$, then the Hilbert function of P/I is
  either $\mathrm{HF}_{P/I} = 1, 2, 3, 4, 4, \ldots$ or $\mathrm{HF}_{\mathrm{P/I}} = 1, 3, 4, 4, \ldots$.

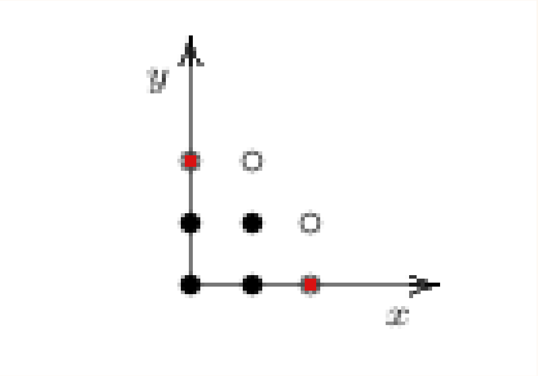
- The difference function is
  either $\mathrm{HF}_{P/I} = 1, 1, 1, 1, 0, \ldots$ or $\mathrm{HF}_{\mathrm{P/I}} = 1, 2, 1, 0, \ldots$.

- What are the possible good bases?

Good bases







Border Basis Schemes

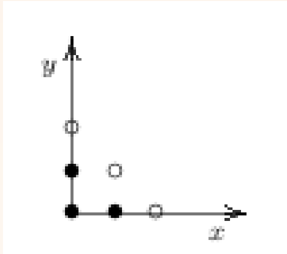
Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal in $\mathbb{T}^n$, and let $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ be its border.

Definition (The Border Basis Scheme)

Let $\{c_{ij} \mid 1 \leq i \leq \mu,\ 1 \leq j \leq \nu\}$ be a set of further indeterminates.

1. The generic $\mathcal{O}$-border prebasis is the set of polynomials $G = \{g_1, \ldots, g_\nu\}$ in $Q = K[x_1, \ldots, x_n, c_{11}, \ldots, c_{\mu\nu}]$ given by $g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$.
2. For $k = 1, \ldots, n$, let $\mathcal{A}_k \in \mathrm{Mat}_\mu(K[c_{ij}])$ be the k^{th} formal multiplication matrix associated to G. Then the affine scheme $\mathbb{B}_\mathcal{O} \subseteq K^{\mu\nu}$ defined by the ideal $I(\mathbb{B}_\mathcal{O})$ generated by the entries of the matrices $\mathcal{A}_k\mathcal{A}_\ell - \mathcal{A}_\ell\mathcal{A}_k$ with $1 \leq k < \ell \leq n$ is called the $\mathcal{O}$-border basis scheme.

The three points 1



- We want to represent all zero-dimensional subschemes of $\mathbb{A}^2_K$ such that the residue classes of the elements in $\{1, x, y\}$ form a basis of their coordinate ring, when viewed as a K-vector space.

- In particular, the elements x^2, xy, y^2 should be expressed modulo I as linear combination of the elements in $\{1, x, y\}$.

- In other words, the ideal I must contain three polynomials

$$
\begin{aligned}
g_1 &= x^2 - c_{11} - c_{21}x - c_{31}y \,, \\
g_2 &= xy - c_{12} - c_{22}x - c_{32}y \,, \\
g_3 &= y^2 - c_{13} - c_{23}x - c_{33}y
\end{aligned}
$$

for suitable values of the coefficients c_{ij}.

The three points 2

- $\{1, x, y\}$ is an order ideal of monomials and the complementary monomial ideal is generated by $\{x^2, xy, y^2\}$.
- If σ is a degree-compatible term ordering, for instance $\sigma = \texttt{DegRevLex}$ then $\mathsf{LT}_\sigma(g_1) = x^2$, $\mathsf{LT}_\sigma(g_2) = xy$, $\mathsf{LT}_\sigma(g_3) = y^2$, no matter which values are taken by the c_{ij} 's.
- We know that $\dim_K(P/I) = \dim_K(P/\mathsf{LT}_\sigma(I))$ and we want that this number is 3 .
- On the other hand, $\dim_K(P/(x^2, xy, y^2)) = 3$, so we want that $\mathsf{LT}_\sigma(I) = (x^2, xy, y^2)$. In other words we want to impose that $\{g_1, g_2, g_3\}$ is a σ -Gröbner basis of I .
- How do we do that? There are two fundamental syzygies of the power products x^2, xy, y^2 namely $(-y, x, 0)$ and $(0, -y, x)$.

The three points 3

- Imposing that $\{g_1, g_2, g_3\}$ is the reduced σ-Gröbner basis of I is equivalent to imposing that some polynomial expressions in the c_{ij} are zero.
- Let J be the ideal of $K[c_{11}, \ldots, c_{33}]$ generated by these polynomials. We check with CoCoA that there is an isomorphism between $K[c_{11}, \ldots, c_{33}]/J$ and $K[c_{21}, c_{31}, c_{22}, c_{32}, c_{23}, c_{33}]$.
- The conclusion is that our family is parametrized by an affine space $\mathbb{A}^6_K$. All such ideals are generated by $\{g_1, g_2, g_3\}$ where

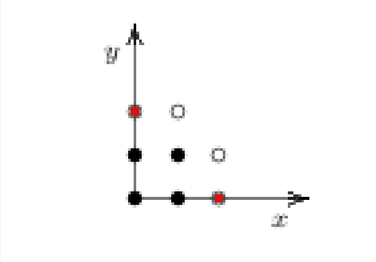
$$g_1 = x^2 - (-c_{21}c_{32} - c_{31}c_{33} + c_{22}c_{31} + c_{32}^2) - c_{21}x - c_{31}y$$
$$g_2 = xy - (c_{21}c_{22} + c_{31}c_{23} - c_{22}c_{21} - c_{32}c_{22}) - c_{22}x - c_{32}y$$
$$g_3 = y^2 - (c_{22}^2 + c_{32}c_{23} - c_{23}c_{21} - c_{33}c_{22}) - c_{23}x - c_{33}y$$

and the parameters can vary freely.

Q1: Consider the process of imposing that $\{g_1, g_2, g_3\}$ is the reduced σ-Gröbner basis of I. Is it canonical?

Q2: Can we do the same for every basis $\mathcal{O}$?

The Four Points



Let $\mathcal{O} = \{1, x, y, xy\}$. We observe that $t_1 = 1$, $t_2 = x$, $t_3 = y$, $t_4 = xy$, $b_1 = x^2$, $b_2 = y^2$, $b_3 = x^2y$, $b_4 = xy^2$. Let $\sigma = \texttt{DegRevLex}$, so that $x >_\sigma y$.

$$\begin{aligned} g_1 &= x^2 - c_{11}1 - c_{21}x - c_{31}y - c_{41}xy \\ g_2 &= y^2 - c_{12}1 - c_{22}x - c_{32}y - \mathbf{c_{42}}xy \\ g_3 &= x^2y - c_{13}1 - c_{23}x - c_{33}y - c_{43}xy \\ g_4 &= xy^2 - c_{14}1 - c_{24}x - c_{34}y - c_{44}xy \end{aligned}$$

- Then necessarily $\mathbf{c_{42} = 0}$ (the linear space), so that g_2 is replaced by

$$g_2^* = y^2 - c_{12}1 - c_{22}x - c_{32}y$$

- If we do the Gröbner computation via critical pairs, as we did before, we get a seven-dimensional scheme $\mathbb{Y}$.
- If we use Mourrain criterion to get the border basis scheme we get an eigth-dimensional scheme $\mathbb{X}$ such that $\mathbb{Y}$ is an hyperplane section.

Philosophy

- A border basis of an ideal of points I in P is intrinsically related to a basis $\overline{\mathcal{O}}$ of the quotient ring.
- If we move the points slightly, $\overline{\mathcal{O}}$ is still a basis of the perturbed ideal $\tilde{I}$, since the evaluation matrix of the elements of $\mathcal{O}$ at the points has determinant different from zero.
- Moving the points moves the border basis, and the movement traces a path inside the border basis scheme.
- On the other hand, if we perturb the equations of the border basis, in general the multiplication matrices almost commute, but most likely the new ideal is the unit ideal.

Seven Generic Points

- Let $\mathbb{X} = \{p_1, p_2, \ldots, p_7\}$ be a set of seven generic points in the affine plane and let $\mathcal{O} = \{1, x, y, x^2, y^2, x^3, y^3\}$.
- Distracting the complementary ideal of $\mathcal{O}$, i.e. (x^4, xy, y^4) yields the ideal $I = (x(x-1)(x-2)(x-3), xy, y(y-1)(y-2)(y-3))$ of seven distinct points such that $\mathcal{O}$ is a basis modulo I .
- To be a basis modulo the ideal of seven distinct points is an open condition. It is not empty by the preceding item.
- Therefore if $\mathbb{X}$ is a generic set of seven points and $I_{\mathbb{X}}$ is its defining ideal, then $\mathcal{O}$ is a basis modulo $I_{\mathbb{X}}$.

Seven Generic Points

- Let $J = I_{\mathbb{X}}^{\text{hom}}$ be the homogenization of $I_{\mathbb{X}}$ with respect to z. We know that $J_{z=1} = I_{\mathbb{X}}$ and that $J_{z=0} = \text{DF}(I_{\mathbb{X}})$.

- CLAIM: The family $K[z] \longrightarrow P[z]/J$ is flat.

  Proof: Standard Gröbner basis theory.

- CLAIM: $\mathcal{O}$ is a basis modulo J_z for every $z \neq 0$.

  Proof: The evaluation matrix of $\mathcal{O}$ at $\mathbb{X}$ has determinant D different from zero. $I_{\mathbb{X}}^{\text{hom}}$ can be obtained by intersecting the homogenization of the seven ideals of the points. i.e $(x_1 - a_{i1}z, x_2 - a_2 z, \ldots, x_n - a_{in}z)$ for $i = 1, \ldots, 7$. The determinant of the evaluation matrix at the points $(a_{i1}z, a_{i2}z, \ldots, a_{in}z)$ is $z^{12}D$, and hence different from zero for every value of $z \neq 0$.

- CLAIM: $\mathcal{O}$ is not a basis modulo $J_{z=0}$.

  Proof: The Hilbert function of the coordinate ring of seven generic points in the projective plane is $1, 3, 6, 7, 7, \ldots$ The Hilbert function of $K[x, y]/\text{DF}(I_{\mathbb{X}})$ is the difference function $1, 2, 3, 1, 0, 0 \ldots$. On the other hand the Hilbert function of $\mathcal{O}$ is $1, 2, 2, 2, 0, 0 \ldots$.

The Gröbner Scheme and the Universal Family

- Gröbner basis schemes and their associated universal families can be viewed as weighted projective schemes.
- Gröbner basis schemes can be obtained as sections of border basis schemes with suitable linear spaces.
- The process of construction Gröbner basis schemes via Buchberger's Algorithm turns out to be canonical.
- Let $\mathcal{O}$ be an order ideal and σ a term ordering on $\mathbb{T}^n$. If the order ideal $\mathcal{O}$ is a σ -cornercut then there is a natural isomorphism of schemes between $G_{\mathcal{O},\sigma}$ and $B_{\mathcal{O}}$.

Border Bases and Hilbert Schemes

Here we collect some basic observations about border basis schemes in relation with Hilbert schemes.
A reference is
E. Miller, B. Sturmfels: Combinatorial Commutative Algebra, Graduate Texts in Mathematics 277, Springer 2005.

- $\mathbb{B}_{\mathcal{O}}$ can be embedded as an open affine subscheme of the Hilbert scheme which parametrizes subschemes of $\mathbb{A}^n$ of length μ .
- There is an irreducible component of $\mathbb{B}_{\mathcal{O}}$ of dimension $n\,\mu$ which is the closure of the set of radical ideals having an $\mathcal{O}$ -border basis.
- The border basis scheme is in general reducible (see the well-known example by Iarrobino).
- In the case $n = 2$ more precise information is available: for instance, it is known that $\mathbb{B}_{\mathcal{O}}$ is reduced, irreducible and smooth of dimension 2μ . Recently M. Huibregtse showed that they are a complete intersection.

Open Problem and the Example by Iarrobino

The scheme $\mathbb{G}_{\mathcal{O},\sigma}$ is connected since it is a quasi-cone, and hence all its points are connected to the origin.

We know the precise relation between the two schemes $\mathbb{G}_{\mathcal{O},\sigma}$ and $\mathbb{B}_{\mathcal{O}}$. However, the problem of the connectedness of $\mathbb{B}_{\mathcal{O}}$ is still open.

Example

Iarrobino in 1972 proves that Hilbert schemes need not be irreducible. In particular, he produces an example which can easily be explained using homogeneous border basis schemes. Let $\mathcal{O}$ be an order ideal in $\mathbb{T}^3$ consisting of all terms of degree ≤ 6 and 18 terms of degree seven. A special subscheme of $\mathbb{B}_{\mathcal{O}}$, called $\mathbb{B}_{\mathcal{O}}^{\text{hom}}$, is isomorphic to an affine space of dimension 324. In particular, it follows that $\dim(\mathbb{B}_{\mathcal{O}}) \geq 324$. On the other hand, the irreducible component of $\mathbb{B}_{\mathcal{O}}$ containing the points corresponding to reduced ideals has dimension $3 \cdot \mu = 3 \cdot 102 = 306$.

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PART 3

Computing Ideals of Points

by John Abbott

PART 4

Using Points for Mathematical Models

Points and Statistics

The following definition originated in a special branch of Statistics called Design of Experiments (for short DoE).

Definition

Let $\ell_i \geq 1$ for $i = 1, \ldots, n$ and $D_i = \{a_{i1}, a_{i2}, \ldots, a_{i\ell_i}\}$ with $a_{ij} \in K$.

- The affine point set $D = D_1 \times \cdots \times D_n \subseteq K^n$ is called the full design on $(D_1, \ldots, D_n)$ with levels $\ell_1, \ldots, \ell_n$.
- The polynomials $f_i = (x_i - a_{i1}) \cdots (x_i - a_{i\ell_i})$ with $i = 1, \ldots, n$ generate the vanishing ideal $\mathcal{I}(D)$ of D . They are called the canonical polynomials of D .

Proposition

- *For any term ordering σ on $\mathbb{T}^n$, the canonical polynomials are the reduced σ -Gröbner basis of $\mathcal{I}(D)$.*
- *The order ideal $\mathcal{O}_D = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid 0 \leq \alpha_i < \ell_i \text{ for } i = 1, \ldots, n\}$ is canonically associated to D and represents a K -basis of $P/\mathcal{I}(D)$.*
- *We have $\mathcal{I}(\mathbb{X}) = (D_\pi(x_1^{r_1}), \ldots, D_\pi(x_n^{r_n}))$*
- *For every affine point set $\mathbb{Y}$, there is a unique minimal full design containing $\mathbb{Y}$.*

Points and Statistics II

- The main task is to identify an unknown function $\bar{f} : D \longrightarrow K$ called the model.
- In general it is not possible to perform all experiments corresponding to the points in D and measuring the value of $\bar{f}$ each time.
- A subset F of a full design D is called a fraction.
- We want to choose a fraction $F \subseteq D$ that allows us to identify the model if we have some extra knowledge about the form of $\bar{f}$.
- In particular, we need to describe the order ideals whose residue classes form a K -basis of $P/\mathcal{I}(F)$. Statisticians express this property by saying that such order ideals are identified by F .

A Classical Example

A number of similar chemical plants had been successfully operating for several years in different locations.
In a newly constructed plant the filtration cycle took almost twice as long as in the older plants.
Seven possible causes of the difficulty were considered by the experts.

1. The water for the new plant was different in mineral content.
2. The raw material was not identical in all respects to that used in the older plants.
3. The temperature of filtration in the new plant was slightly lower than in the older plants.
4. A new recycle device was absent in the older plants.
5. The rate of addition of caustic soda was higher in the new plant.
6. A new type of filter cloth was being used in the new plant.
7. The holdup time was lower than in the older plants.

A Classical Example II

- These causes lead to seven variables $x_1, \ldots, x_7$. Each of them can assume only two values, namely *old* and *new* which we denote by 0 and 1 , respectively.
- All combinations of these values form the full design $D = \{0,1\}^7 \subseteq \mathbb{A}^7(\mathbb{Q})$. Its vanishing ideal is $\mathcal{I}(D) = (x_1^2 - x_1,\, x_2^2 - x_2, \ldots,\, x_7^2 - x_7)$ in $\mathbb{Q}[x_1, \ldots, x_7]$.
- Our task is to identify an unknown function $\bar{f} : D \longrightarrow K$, the length of a filtration cycle. It is the model which has to be computed or optimized.
- In order to fully identify it, we would have to perform $128 = 2^7$ cycles. This is impracticable since it would require too much time and money.
- On the other hand, suppose for a moment that we had conducted all experiments and the result was $\bar{f} = a + b\,x_1 + c\,x_2$ for some $a, b, c \in \mathbb{Q}$. Had we known in advance that $\bar{f}$ is given by a polynomial having only three unknown coefficients, we could have identified them by performing only *three* suitable experiments!
- However, *a priori* one does not know that the answer has the shape indicated above. One has to make some guesses, perform well-chosen experiments, and possibly modify the guesses until the process yields the desired answer.
- In the case of the chemical plant, it turned out that only x_1 and x_5 were relevant for identifying the model.

A Proposition

Proposition

The following conditions are equivalent.

- *The order ideal* $\mathcal{O}$ *is identified by the fraction* F *.*
- *The vanishing ideal* $\mathcal{I}(F)$ *has an* $\mathcal{O}$ *-border basis.*
- *We have* $\mu = \nu$ *and* $\det(t_i(p_j)) \neq 0$ *.*

- How can we choose the fraction F such that the matrix of coefficients is invertible?
- In other words, given a full design D and an order ideal $\mathcal{O} \subseteq \mathcal{O}_D$, which fractions $F \subseteq D$ have the property that the residue classes of the elements of $\mathcal{O}$ are a K -basis of $P/\mathcal{I}(F)$?
- We call this the inverse problem.

An Algorithm

We assume that D is minimal, in the sense that $\mathcal{O}_\mathcal{D}$ is the minimal order ideal which contains $\mathcal{O}$.

1. Let $C = \{f_1, \ldots, f_n\}$ be the set of canonical polynomials of D , where $\mathrm{LT}_\sigma(f_i) = x_i^{\ell_i}$ for any term ordering σ .
2. Decompose $\partial\mathcal{O}$ into two subsets $\partial\mathcal{O}_1 = \{x_1^{\ell_1}, \ldots, x_n^{\ell_n}\}$ and $\partial\mathcal{O}_2 = \partial\mathcal{O} \setminus \partial\mathcal{O}_1$.
3. Let $\eta = \#(\partial\mathcal{O}_2)$. For $i = 1, \ldots, \eta$ and $j = 1, \ldots, \mu$, introduce new indeterminates z_{ij} .
4. For every $b_k \in \partial\mathcal{O}_2$, let $g_k = b_k - \sum_{j=1}^{\mu} z_{kj} t_j \in K(z_{ij})[x_1, \ldots, x_n]$.
5. Let $G = \{g_1, \ldots, g_\eta\}$ and $H = G \cup C$. Let $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be the formal multiplication matrices associated to the $\mathcal{O}$ -border prebasis H .
6. Let $\mathcal{I}(\mathcal{O})$ be the ideal in $K[z_{ij}]$ generated by the entries of the matrices $\mathcal{M}_i\mathcal{M}_j - \mathcal{M}_j\mathcal{M}_i$ for $1 \le i < j \le n$.

Then $\mathcal{I}(\mathcal{O})$ is a zero-dimensional ideal in $K[z_{ij}]$ whose zeros are in 1-1 correspondence with the solutions of the inverse problem, i.e. with fractions $F \subseteq D$ such that $\mathcal{O}$ represents a K -basis of $P/\mathcal{I}(F)$.

Parametrizations, hyperplane sections, Hough transforms

M.C. Beltrametti, L. Robbiano,

An algebraic approach to Hough transforms

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L. Robbiano,

Parametrizations, hyperplane sections, Hough transforms

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Abstract

- The main purpose of this lecture is to show how to extend the classical Hough Transforms (HT for short) to general algebraic schemes.
- Initially, following some ideas presented in Beltrametti-Robbiano, we develop an algorithmic approach to decide whether a parametrization is rational.
- Then we concentrate on hyperplane sections of algebraic families, and present some new theoretical results for determining when a given parametrization via reduced Gröbner bases passes to the quotient, and viceversa.
- As a by-product we hint at a promising technique for computing implicitizations quickly.
- Finally, we apply the results about hyperplane sections to families of algebraic schemes and their HTs.
- These last results hint at the possibility of reconstructing external and internal surfaces of human organs from the parallel cross-sections obtained by tomography.

History

- The Hough transform (or transformation) is a technique mainly used in image analysis and digital image processing.
- It was introduced by P.V.C. Hough in 1962 in the form of a patent.
- Its intended application was in physics for detection of segments and arcs in the photographs obtained in particle detectors.
- Many elaborations and refinements of this method have been investigated since.
- The main tool to achieve such result is a voting procedure which is used in a parameter space.
- Let us see how it works.

Detecting Aligned Points

- Suppose we want to detect aligned points in a given picture.
- Let us represent a straight line as $y = ax + b$ (not the best representation!).
- Let us subdivide the picture into small cells (points).
- For every cell/point $p = (x_0, y_0)$ a straight line containing it is such that $y_0 = ax_0 + b$.
- MAIN IDEA: $y_0 = ax_0 + b$ represents a straight line in the space of parameters.
- If the space of parameters is subdivided in cells, we assign the vote 1 for every cell hit by the line $y_0 = ax_0 + b$.
- We repeat this process for every point/cell in the picture and keep adding votes to the cells in the parameter space.
- If, say, a cell (a_0, b_0) gets a lot of votes, it means that many points in the picture lie on the line $y = a_0x + b_0$!!!

Transition to Algebraic Geometry

If we want to detect more complicated algebraic varieties we need a suitable parameter space. Let us see how it works

- $\Phi : \mathcal{F} \longrightarrow \mathbb{A}^m_K$ is a dominant family of sub-schemes of $\mathbb{A}^n_K$.
  It corresponds to a K -algebra homomorphism $\varphi : K[\mathbf{a}] \longrightarrow K[\mathbf{a}, \mathbf{x}]/I(\mathbf{a}, \mathbf{x})$.
- If we fix $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{A}^m_K$ and get the fiber $\mathrm{Spec}(K[\boldsymbol{\alpha}, \mathbf{x}]/I(\boldsymbol{\alpha}, \mathbf{x}))$, hence a special member of the family. We consider $I(\boldsymbol{\alpha}, \mathbf{x})$ as an ideal in $K[\mathbf{x}]$. With this convention we denote the scheme $\mathrm{Spec}(K[\mathbf{x}]/I(\boldsymbol{\alpha}, \mathbf{x}))$ by $\mathbb{X}_{\boldsymbol{\alpha}, \mathbf{x}}$.
- On the other hand, there exists another morphism $\Psi : \mathcal{F} \longrightarrow \mathbb{A}^n_K$ which corresponds to the K -algebra homomorphism $\psi : K[\mathbf{x}] \longrightarrow K[\mathbf{a}, \mathbf{x}]/I(\mathbf{a}, \mathbf{x})$.
  It is not necessarily dominant.
- If we fix $p = (\xi_1, \dots, \xi_n) \in \mathbb{A}^n_K$, we get the fiber $\mathrm{Spec}(K[\mathbf{a}, p]/I(\mathbf{a}, p))$.
  We consider $I(\mathbf{a}, p)$ as an ideal in $K[\mathbf{a}]$. With this convention we denote the scheme $\mathrm{Spec}(K[\mathbf{a}]/I(\mathbf{a}, p))$ by $\Gamma_{\mathbf{a}, p}$.

The Hough Transform (HT)

Definition

Let $p = (\xi_1, \ldots, \xi_n) \in A^n_K$. Then the scheme $\Gamma_{\mathbf{a},p}$ is said to be the Hough transform of the point p (with respect to the family Φ), and denoted by H_p .

Remark: It can be empty.

Definition

Let σ be a degree-compatible term ordering, and let $G(\mathbf{a}, \mathbf{x})$ be the reduced σ -Gröbner basis of $I(\mathbf{a}, \mathbf{x})K(\mathbf{a})[\mathbf{x}]$. We let $d_\sigma(\mathbf{a})$ be the l.c.m. of the denominators of the coefficients in G , and call it the σ -denominator of Φ.
We say that $\mathcal{U}_\sigma = \mathbb{A}^m \setminus \{d_\sigma(\mathbf{a}) = 0\}$ is the σ -free set of the family $\mathcal{F}$, that $\Phi_{|d_\sigma(\mathbf{a})} : \Phi^{-1}(\mathcal{U}_\sigma) \longrightarrow \mathcal{U}_\sigma$ is the σ -free restriction of Φ, and that $G_\sigma(\mathbf{a}, \mathbf{x})$ is the universal reduced σ -Gröbner basis of $\mathcal{F}$.
Finally, we say that NCC_G is the non-constant coefficient list of $G(\mathbf{a}, \mathbf{x})$.

An Example and a Theorem

Example

Let $\mathbf{a} = (a, b)$, let $\mathbf{x} = (x, y)$, and let $\mathcal{F} = \mathrm{Spec}(K[\mathbf{a}, \mathbf{x}]/F_{\mathbf{a},\mathbf{x}})$, where

$$F_{\mathbf{a},\mathbf{x}} = y(x - ay)^2 - b(x^4 + y^4)$$

Then let $\boldsymbol{\alpha} = (1, 1)$. The corresponding fiber in $\mathbb{A}^2_K$ is $C_{(1,1),\,\mathbf{x}}$ which is defined by the polynomial $F_{(1,1),\,\mathbf{x}} = y(x - y)^2 - (x^4 + y^4)$.

The point $p = (0, 1)$ belongs to $C_{(1,1),\,\mathbf{x}}$ and it corresponds to the curve $\Gamma_{\mathbf{a},(0,1)}$ which is defined by the polynomial $F_{\mathbf{a},(0,1)} = a^2 - b$.

We have $\Gamma_{\mathbf{a},(0,1)} = \mathrm{H}_{(0,1)}$ i.e. the parabola $\Gamma_{\mathbf{a},(0,1)}$ is the HT of $(0, 1)$.

Theorem

The correspondence between $\{\mathbb{X}_{\alpha,\mathbf{x}} \mid \alpha \in \mathcal{U}_\sigma\}$ *and* NCC_G *which is defined by sending* $\mathbb{X}_{\alpha,\mathbf{x}}$ *to* $\mathrm{NCC}_G(\alpha)$ *is bijective.*

Hough Regularity

If $\cap_{p \in \mathbb{X}_{\boldsymbol{\alpha},\mathbf{x}}} \Gamma_{\mathbf{a},p} = \{\boldsymbol{\alpha}\}$ for all $\boldsymbol{\alpha} \in \mathcal{U}_\sigma$, we say that $\Phi_{|d_\sigma(\mathbf{a})}$ is Hough σ -regular.

Proposition

The following conditions are equivalent.

(a) *The morphism* $\Phi_{|d_\sigma(\mathbf{a})}$ *is Hough* σ *-regular.*

(b) *For all* $\boldsymbol{\alpha}, \boldsymbol{\eta} \in \mathcal{U}_\sigma$ *, the equality* $\mathbb{X}_{\boldsymbol{\alpha},\mathbf{x}} = \mathbb{X}_{\boldsymbol{\eta},\mathbf{x}}$ *implies* $\boldsymbol{\alpha} = \boldsymbol{\eta}$.

Corollary

The following conditions are equivalent.

(a) *The morphism* $\Phi_{|d_\sigma(\mathbf{a})}$ *is Hough* σ *-regular.*

(b) *The map* $\mathcal{U}_\sigma \longrightarrow NCC_{G_\sigma}$ *which sends* α *to* $NCC_{G_\sigma}(\alpha)$ *is one-to-one.*

Rationality: When is this map one-to-one?

Definition

In $K[\mathbf{a},\mathbf{e}]$ the ideal $(p_1(\mathbf{a})d_1(\mathbf{e}) - p_1(\mathbf{e})d_1(\mathbf{a}), \dots, p_s(\mathbf{a})d_s(\mathbf{e}) - p_s(\mathbf{e})d_s(\mathbf{a}))$ is called the ideal of doubling coefficients of $\mathcal{P}$, and the ideal $(a_1 - e_1, \dots, a_m - e_m)$ is called the diagonal ideal.

Theorem

Let K be algebraically closed, let $I(DC_G)$ be the ideal of doubling coefficients of G, let $I(\Delta)$ be the diagonal ideal, and let $S(\Delta)$ be the saturation of $I(DC_G)$ with respect to $I(\Delta)$. Then the following conditions are equivalent.

(a) *The morphism $\Phi_{|d_\sigma(\mathbf{a})}$ is Hough σ-regular.*

(b) *The ideal $I(\Delta)$ is contained in the radical of the ideal $I(DC_G)$.*

(c) *The ideal $I(\Delta)$ coincides with the radical of the ideal $I(DC_G)$.*

(d) *We have $S(\Delta) = (1)$.*

An Example

Example

Let $\Phi : \mathcal{F} \longrightarrow \mathbb{A}^2$ be defined parametrically by

$$x = a_1 u^4, \;\; y = u^5, \;\; z = a_2 u^6$$

By eliminating u we get generators of the ideal $I(\mathbf{a}, \mathbf{x})$, and if $\sigma = \texttt{DegRevLex}$, the reduced σ -Gröbner basis of $I(\mathbf{a}, \mathbf{x})K(\mathbf{a})[\mathbf{x}]$ is

$$G = (y^2 - \frac{1}{a_1 a_2} xz, \;\; x^3 - \frac{a_1^3}{a_2^2} z^2\})$$

CoCoA Code

```
R ::= QQ[a[1..2]];
S ::= QQ[t, a[1..2], e[1..2]];
K := NewFractionField(R);
Use P ::= K[x,y,z,u];
ID:=Ideal(x-a[1]*u^4, y-u^5, z-a[2]*u^6);
E:=Elim([u],ID);
RGB := ReducedGBasis(E);
NCC := NonConstCoefficients(RGB);RGB;
Use S;
IDelta := ideal([a[i]-e[i] | i In 1..2]);
IDC := IdealOfDoublingCoefficients(S, NCC, "a", "e", "t");
IsInRadical(IDelta, IDC);
--false
```

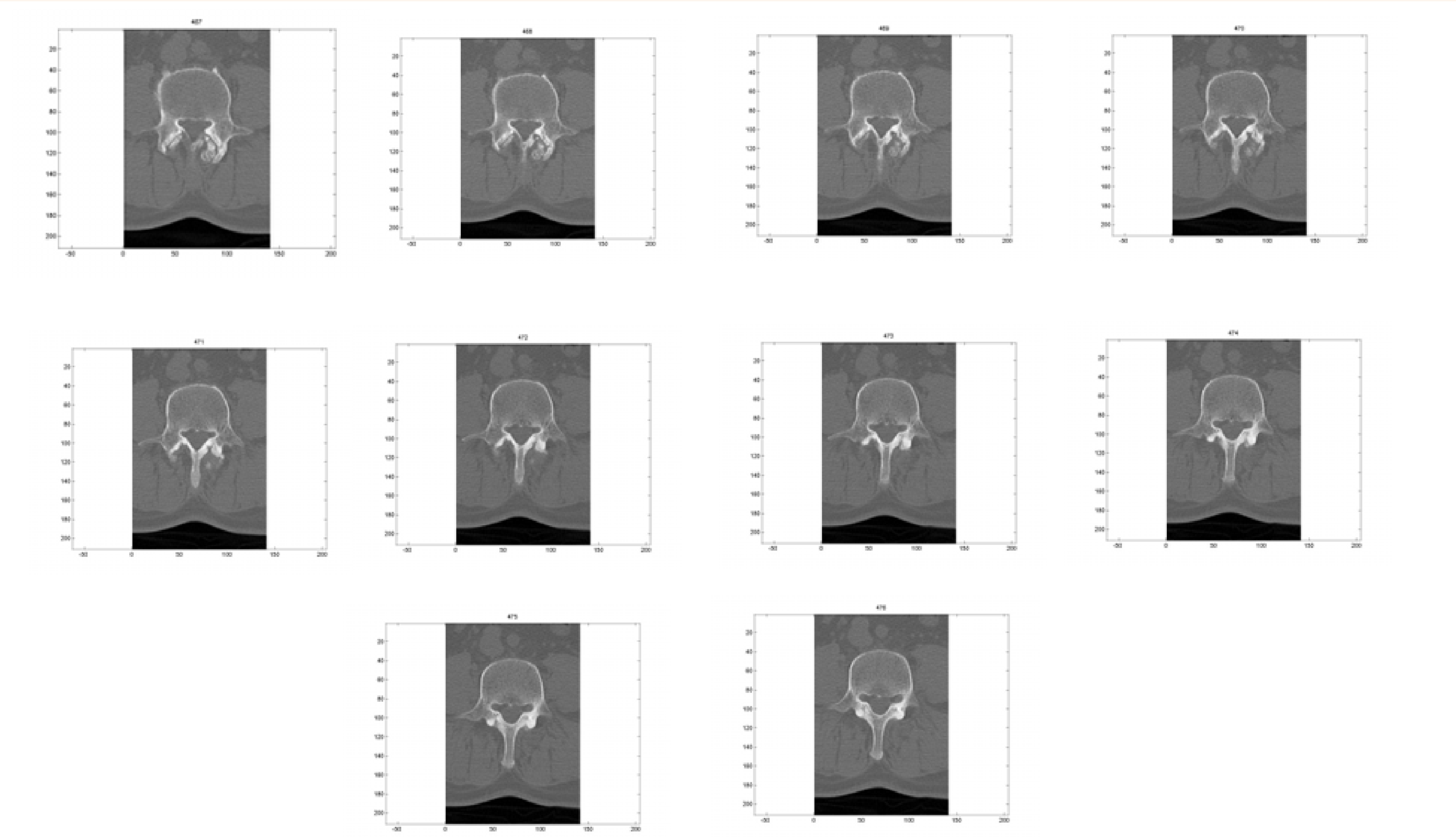
The family is not Hough-regular as the CoCoA-code showed.

We have $\mathrm{NCC}_G = (-\frac{1}{a_1 a_2}, -\frac{a_1^3}{a_2^2})$.

If ε is the primitive fifth root of unity, then $\mathrm{NCC}_G(\varepsilon^2, \varepsilon^3) = \mathrm{NCC}_G(-1, -1)$.

Hyperplane Sections

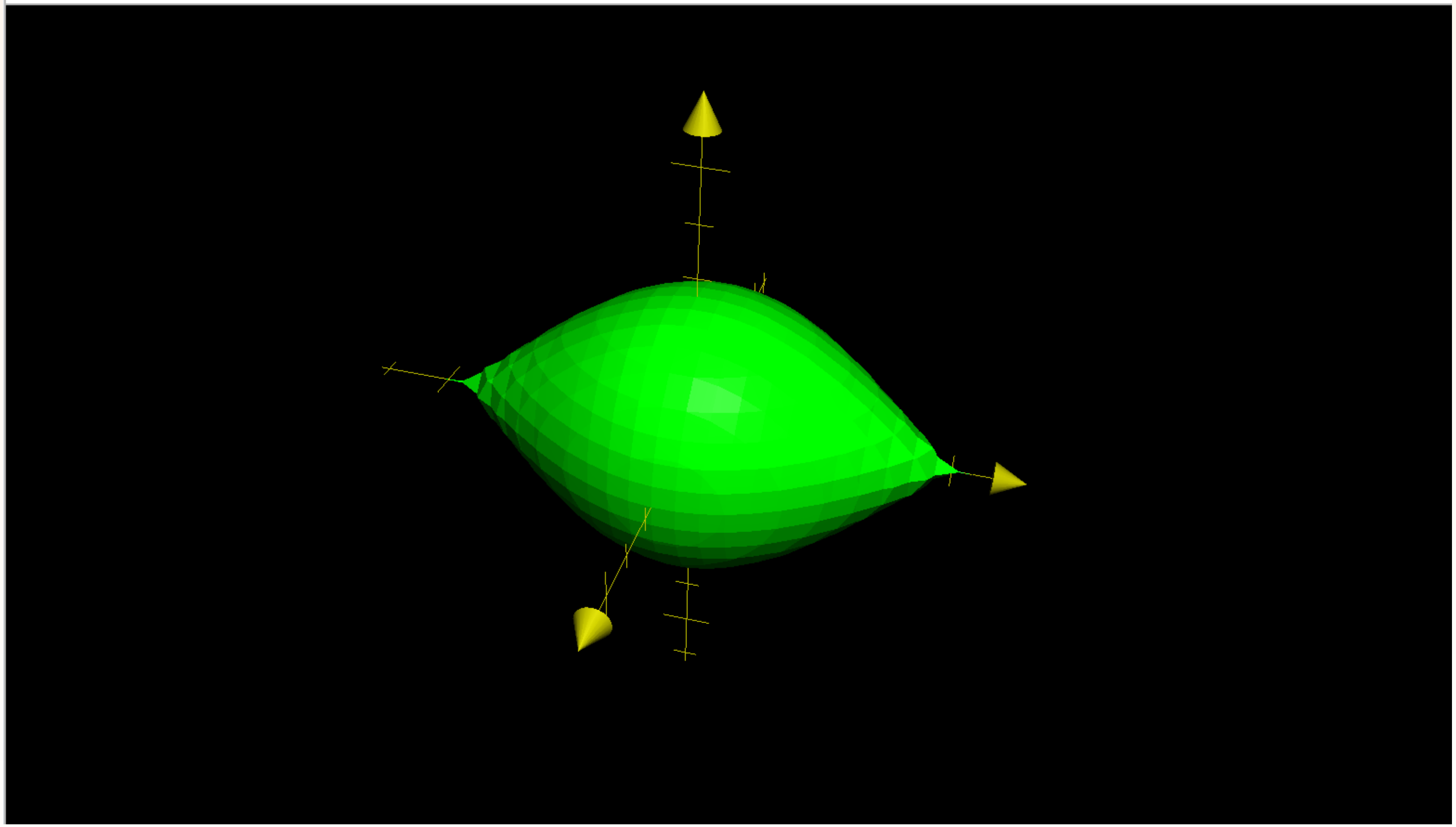
Lumbar Vertebrae



The suggested problem is to recover the reduced Gröbner basis of a scheme via the reduced Gröbner bases of its hyperplane sections.

Zitrus by Herwig Hauser for Imaginary

$$(x)^2 + (z)^2 - 3.5 \cdot ((y+0.4))^3 \cdot (0.43-y)^3 = 0$$



Exact Reconstruction of Hypersurfaces

HyperplaneSections.cocoa5

Hyperplane Sections 2

ASSUMPTIONS: Fix an element $i \in \{1, \dots, n\}$ and a linear form $\ell = \sum_{j \geq i} c_j x_j$. If $\gamma \in K$, let $L = x_i - (\ell + \gamma)$ and then identify $P/(L)$ with the polynomial ring $\hat{P} = K[x_1, \dots, x_{i-1}, x_{i+1}, \dots, n]$ via the isomorphism induced by $\varphi_L(x_i) = \ell + \gamma$, $\varphi_L(x_j) = x_j$ for $j \neq i$.

NOTATION: If σ is a term ordering on $\mathbb{T}^n$, we call $\hat{\sigma}$ the restriction of σ to the monoid $\mathbb{T}_{\hat{\imath}} = \mathbb{T}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, n)$.

Theorem

Under the above assumptions let σ be a term ordering such that $x_1 >_\sigma x_2 >_\sigma \cdots >_\sigma x_n$, let I be an ideal in P , let $G = \{g_1, \dots, g_s\}$ be a σ -Gröbner basis of I , and assume that x_i does not divide zero modulo $\mathsf{LT}_\sigma(I)$.

(a) *The set $\varphi_L(G) = \{\varphi_L(g_1), \dots, \varphi_L(g_s)\}$ is a $\hat{\sigma}$ -Gröbner basis of $\varphi_L(I)$.*

(b) *If G is the reduced σ -Gröbner basis of I , then $\varphi_L(G)$ is the reduced $\hat{\sigma}$ -Gröbner basis of $\varphi_L(I)$.*

(c) *The ideals $\mathsf{LT}_\sigma(I)$ and $\mathsf{LT}_{\hat{\sigma}}(\varphi_L(I))$ have the same minimal set of monomial generators.*

Lifting

Theorem

Under the above assumptions let I *be an ideal in* P *such that* L *does not divide zero modulo* I*, let* $\widehat{G} = \{\hat{g}_1, \dots, \hat{g}_s\}$ *be a* $\hat{\sigma}$*-Gröbner basis of* I_L*, and let* $G = \{g_1, \dots, g_s\} \subset I$ *such that* $\varphi(g_i) = \hat{g}_i$ *and* $\mathrm{LT}_\sigma(g_i) = \mathrm{LT}_{\hat{\sigma}}(\hat{g}_i)$ *for all* i*.*

(a) *The set* $G = \{g_1, \dots, g_s\}$ *is a* σ*-Gröbner basis of* I*.*

(b) *If* $\widehat{G}$ *is the reduced* $\hat{\sigma}$*-Gröbner basis of* J*, then* G *is the reduced* σ*-Gröbner basis of* I*.*

Theorem

Under the above assumptions let I *be an ideal in* P*, let* $\gamma_1, \dots, \gamma_N$ *be elements in* K*, let* $L_k = x_i - (\ell + \gamma_k)$*.*

(a) *If* $\gamma_1, \dots, \gamma_N$ *are generic, the reduced* $\hat{\sigma}$*-Gröbner bases of the ideals* $\varphi_{L_k}(I)$ *share the same number of elements and leading terms, say* $t_1, \dots, t_s$*.*

(b) *If* $N \gg 0$*, at least one of the* L_k *does not divide zero modulo* I*.*

(c) *Let* $g_1, \dots, g_s$ *be common liftings of the corresponding polynomials in the reduced* $\hat{\sigma}$*-Gröbner bases of the ideals* $\varphi_{L_k}(I)$*. If* $g_i \in I$ *and* $\mathrm{LT}_\sigma(g_i) = t_i$ *for* $i = 1, \dots, s$ *then* $(g_1, \dots, g_s)$ *is the reduced* σ*-Gröbner basis of* I*.*

Implicitization

QuickImplicit.cocoa5

An Example

Example

Let $\mathcal{F}$ be the sub-scheme of $\mathbb{A}^4$ defined by the ideal

$$I = (x_1^2 - x,\ \ x_1x_2 - x_2,\ \ x_2^2 + a_1a_2x_1 - (a_1 + a_2)x_2)$$

We have the following diagram

$$\begin{array}{ccc} & \mathcal{F} & \\ {}^{\Phi}\swarrow & & \searrow^{\Psi} \\ \mathbb{A}^2_{(a_1,a_2)} & & \mathbb{A}^2_{(x_1,x_2)} \end{array}$$

It is easy to check that $\dim(\mathcal{F}) = 2$, that Φ is dominant while Ψ is not dominant. In particular, the closure of the image of Ψ is the union of the point $(0, 0)$ and the line $x_1 - 1 = 0$.

The above example justifies the reason why in the next proposition we need to consider the image of Ψ .

Dimension

Proposition

Let $\mathbb{X} \subseteq \mathbb{A}^n$ *be the closure of the image of* Ψ *, and let* $\mathbb{Y} \subseteq \mathbb{X}$ *be an irreducible component of* $\mathbb{X}$ *, let* p *be the generic point of* $\mathbb{Y}$ *, and let* $\mathbb{X}_{\boldsymbol{\alpha},\mathbf{x}}$ *be the generic fiber of* Φ *. Then we have*

$$\dim(\Gamma_{\mathbf{a},p}) + \dim(\mathbb{Y}) = \dim(\mathcal{F}) = m + \dim(\mathbb{X}_{\boldsymbol{\alpha},\mathbf{x}})$$

Corollary (Dimension of Hough Transforms)

The following conditions hold.

(a) $\dim(\mathrm{H}_p) = \dim(\mathcal{F}) - \dim(\mathbb{Y}) = m + \dim(\mathbb{X}_{\boldsymbol{\alpha},\mathbf{x}}) - \dim(\mathbb{Y})$.

(b) *If* Ψ *is dominant and* $\dim(\mathcal{F}) = m$ *, then* $\dim(\mathrm{H}_p) = 0$.

(c) *If* $\dim(\mathrm{H}_p) = 0$ *and the generators of* I *are linear polynomials in the parameters* $\mathbf{a}$ *, then* H_p *is a single rational point.*

An Example

Example

Let $\mathcal{F}$ be the sub-scheme of $\mathbb{A}^5$ defined by the ideal I generated by the two polynomials

$$F_1 = (x^2 + y^2)^3 - (a_1 (x^2 + y^2) - a_2 (x^3 - 3xy^2))^2; \quad F_2 = a_1 z - a_2 x.$$

- If we pick a degree-compatible term ordering σ such that $z >_\sigma y >_\sigma x$, then $\mathsf{LT}_\sigma(F_1) = y^6$, $\mathsf{LT}_\sigma(F_2) = z$ if $a_1 \neq 0$, and $\{F_1, \frac{1}{a_1} F_2\}$ is the reduced Gröbner basis of I .
- We have $\mathcal{U}_\sigma = \mathbb{A}^2 \setminus \{a_1 = 0\}$ and we see that $\Phi^{-1}(\mathcal{U}_\sigma) \longrightarrow \mathcal{U}_\sigma$ is free.
- If we perform the elimination of $[a_1, a_2]$ we get the zero ideal, hence also Ψ is dominant, actually surjective.
- Counting dimensions we see that the HT of the points in $\mathbb{A}^3$ are pairs of points.
- For instance, if we pick the point $p = (1, 1, 1)$, its HT is the pair of points $(\frac{1}{\sqrt{2}}, 1)$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

Questions and Problems

1. How to use the CRT to improve implicitization.
2. How to benefit from the dimension formula.
3. How to handle the reconstruction of surfaces, given approximate curves.