`CartierFrobeniusStanleyReisner.cpkg`: a CoCoA package for studying the finite generation of Frobenius algebras of Stanley-Reisner rings

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General setting

Throughout this poster all the rings will be commutative of characteristic p (where p is a positive prime number) and k will be a field of characteristic p.

Let R be a ring and let M be an R-module. A p^e-linear map $M \xrightarrow{\varphi_e} M$ is an additive map that satisfies $\varphi_e(rm) = r^{p^e}\varphi_e(m)$ for each $(r,m) \in R \times M$. The set of all p^e-linear maps is denoted $\mathcal{F}^e(M)$.

Definition (G. Lyubeznik and K. E. Smith, 2001)

The ring of Frobenius operators on M is the graded associative (not necessarily commutative) ring

$$\mathcal{F}(M) := \bigoplus_{e \in \mathbb{N}_0} \mathcal{F}^e(M).$$

They raised the question whether $\mathcal{F}(M)$ is finitely generated as a ring extension of $\mathcal{F}^0(M)$.

Katzman's counterexample

M. Katzman showed in 2010 that if $R := k[[x,y,z]]/I$, where I is the ideal generated by xy and yz, then the R-algebra $\mathcal{F}(E_R)$ is not finitely generated.

Key fact

Let $S := k[[x_1,\dots,x_d]]$, let $I \subset S$ be any ideal and let E_R be the injective hull of the residue field of $R := S/I$. There is an isomorphism of R-algebras

$$\mathcal{F}(E_R) \cong \bigoplus_{e \geq 0} \left(I^{[p^e]} :_S I\right)/I^{[p^e]},$$

where $I^{[p^e]} := \langle x^{p^e} \mid \ x \in I\rangle$.

From now on, let $I \subseteq S$ be a squarefree monomial ideal generated by $\mathrm{x}^{\alpha} := x_1^{\alpha_1}\cdots x_d^{\alpha_d}, \alpha \in \{0,1\}^d$.

Proposition (J. Àlvarez Montaner, A. F. Boix, S. Zarzuela)

Let $I = I_{\alpha_1} \cap \dots \cap I_{\alpha_s}$ be a minimal primary decomposition of I given in terms of face ideals $I_\alpha := \langle x_i \mid \alpha_i \neq 0\rangle$, $\alpha \in \{0,1\}^n$. Then, one has that

$$(I^{[q]} :_S I) = \left(I_{\alpha_1}^{[q]} + (\mathrm{x}^{\alpha_1})^{q-1}\right) \cap \dots \cap \left(I_{\alpha_s}^{[q]} + (\mathrm{x}^{\alpha_s})^{q-1}\right).$$

We must point out that this is a primary decomposition for the colon ideal which was proved by R. Fedder in 1983 for unmixed ideals of any regular local ring of prime characteristic.

Theorem (J. Àlvarez Montaner, A. F. Boix, S. Zarzuela)

The following statements are equivalent.

(i) *$\mathcal{F}(E_R)$ is a finitely generated R-algebra.*
(ii) *$\mathcal{F}(E_R)$ is a principally generated R-algebra.*
(iii) *$(I^{[p^e]} :_S I) = I^{[p^e]} + \langle (x_1\cdots x_d)^{p^e-1}\rangle$ for each $e \in \mathbb{N}_0$.*
(iv) *$(I^{[p]} :_S I) = I^{[p]} + \langle (x_1\cdots x_d)^{p-1}\rangle$.*
(v) *$(I^{[2]} :_S I) = I^{[2]} + \langle x_1\cdots x_d\rangle$.*

How to use it algoritmically?

Require: I squarefree monomial ideal with $\mathrm{ht}(I) > 1$
Ensure: **true** if $\mathcal{F}(E_R)$ is principally generated and **false** otherwise
$J \leftarrow Ideal(MinGens(I))$ {In order to minimalize the number of generators}
$G \leftarrow J^{[2]}$
$G \leftarrow (G : J)$ {Using the primary decomposition founded in the above proposition}
if $Len(G) = Len(J) + 1$ **then**
 return true
else
 return false
end if

In order to avoid the height one case we use Alexander duality through `Frobby`, the software system developed by Bjarke Hammersholt Roune which is integrated inside CoCoA.

Infinitely generated case

Recall that $\mathcal{F}(E_R)$ is an infinitely generated R-algebra if and only if there is an ideal $J_2 \not\subset I^{[2]} + \langle x_1\cdots x_d\rangle$ such that

$$\left(I^{[2]} : I\right) = I^{[2]} + J_2 + \langle x_1\cdots x_d\rangle.$$

When the Frobenius algebra of E_R is infinitely generated we may compute, using our package, the maximum number of generators that each graded piece can reach, using the fact that this upper bound must be reached when I is of separate type.

d	3	4	5	6	7	8	9	10	11	12	13
$\mu(J_2)$	2	4	8	18	26	64	83	210	275	664	875

Here, d is the number of indeterminates of the current polynomial ring. For $d \geq 14$ our algorithm doesn't run.
Now, we introduce the main steps in order to generate the above table.

Step 1 Generate all the squarefree monomial ideals of separate type.
Step 2 For each ideal of separate type compute $\mu(J_2)$. Save these numbers in a list.
Step 3 Return the maximum of the list generated in Step 2.

We say that a squarefree monomial ideal is of **separate type** is there is $s = 1,\dots,d$ and $0 = n_0 < n_1 < \dots < n_s = d$ such that

$$I = \bigcap_{j=1}^{s} \langle x_{n_{j-1}+1},\dots,x_{n_j}\rangle.$$

Finitely generated case

There is a distinguished family of ideals with principally generated Frobenius algebra. Namely, set $S := k[[x_1,\dots,x_d]]/I_{k,d}$, where

$$I_{k,d} := \bigcap_{1 \leq i_1 < \dots < i_k \leq d} \langle x_{i_1},\dots,x_{i_k}\rangle.$$

One may see that $\mathcal{F}(E_S)$ is a principally generated S-algebra.

Underline that $I_{k,d}^{\vee} = I_{d-k+1,d}$. This property allows us to generate so quickly these family of ideals. Here we only treat the case where the number of indeterminates of our current polynomial ring is even.

Ensure: X list with all the ideals $I_{k,d}$ of our current ring in terms of their generators.
$q \leftarrow d/2$
$X \leftarrow [\,]$
for $i = 1$ **to** q **do**
 $Y \leftarrow HIntersectionList(Z)$ {Z is the list which contains all the face ideals of height i}
 $X \leftarrow Concat(X, [Y^{\vee}])$ {We are using standard Alexander duality}
end for
return X

Examples with pure height

It is known that Frobenius algebras of Gorenstein rings are principally generated and the converse also holds for normal varieties.
In the following tables we deal with the case of 4 and 5 indeterminates. Depending on the height we count the number of pure ideals (up to relabeling) having principally generated Frobenius algebra, how many Gorenstein (**Gor**, for short) rings we get among them and the number of ideals having infinitely generated (i.g., for short) Frobenius algebra.

$n = 4$	p.g.	Gor	i.g.
$\mathrm{ht}(I) = 1$	1	1	-
$\mathrm{ht}(I) = 2$	4	2	3
$\mathrm{ht}(I) = 3$	3	1	-
$\mathrm{ht}(I) = 4$	1	1	-

$n = 5$	p.g.	Gor	i.g.
$\mathrm{ht}(I) = 1$	1	1	-
$\mathrm{ht}(I) = 2$	6	2	13
$\mathrm{ht}(I) = 3$	12	2	10
$\mathrm{ht}(I) = 4$	4	1	-
$\mathrm{ht}(I) = 5$	1	1	-

For $d \geq 6$ our algorithm doesn't run.