Involutive Bases V

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- **General Involutive Bases**
- **Basic Algorithms**
- **Pommaret Bases and δ-Regularity**
- **Combinatorial Decompositions and Applications**
- **Syzygy Theory and Applications**
  - Syzygies of involutive bases
  - involutive Schreyer theorem
  - (minimal) free resolutions
  - monomial ideals
  - Castelnuovo-Mumford regularity

Def: $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\} \subset \mathcal{P}^m$, $\mathbf{S} \in \mathcal{P}^s$ *syzygy* of $\mathcal{H}$ $\rightsquigarrow$

$$\mathbf{S} = \sum_{\gamma=1}^{s} S_\gamma \mathbf{e}_\gamma \quad \text{with} \quad \sum_{\gamma=1}^{s} S_\gamma \mathbf{h}_\gamma = 0$$

all syzygyies of $\mathcal{H}$ form *syzygy module* $\operatorname{Syz}(\mathcal{H})$
(by abuse of notation: $\operatorname{Syz}(\mathcal{M})$ for $\mathcal{P}$-module $\mathcal{M} = \langle\mathcal{H}\rangle$)

iteration $\rightsquigarrow$ higher syzygy modules $\operatorname{Syz}_k(\mathcal{H}) = \operatorname{Syz}(\operatorname{Syz}_{k-1}(\mathcal{H}))$

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Def: $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\} \subset \mathcal{P}^m$, $\prec$ term order on $\mathbb{T}(X)^m$ $\rightsquigarrow$
induced *Schreyer order* $\prec_{\mathcal{H}}$ on $\mathbb{T}(X)^s$:

$$s\mathbf{e}_\sigma \prec_{\mathcal{H}} t\mathbf{e}_\tau \iff \begin{array}{l} \left(\operatorname{lt}_\prec(s\mathbf{h}_\sigma) \prec \operatorname{lt}_\prec(t\mathbf{h}_\tau)\right) \vee \\ \left(\left(\operatorname{lt}_\prec(s\mathbf{h}_\sigma) = \operatorname{lt}_\prec(t\mathbf{h}_\tau)\right) \wedge (\tau < \sigma)\right) \end{array}$$

Syzygies of Gröbner Bases

- assume $\mathcal{H}$ *Gröbner basis*
- choose $\mathbf{t}_\alpha = \operatorname{lt}\mathbf{h}_\alpha, \mathbf{t}_\beta = \operatorname{lt}\mathbf{h}_\beta$ with $\mathbf{t}_{\alpha\beta} = \operatorname{lcm}(\mathbf{t}_\alpha, \mathbf{t}_\beta) \neq \mathbf{0}$
- any standard representation of S-“polynomial”

$$\mathbf{S}(\mathbf{h}_\alpha, \mathbf{h}_\beta) = \sum_{\gamma=1}^{s} f_{\alpha\beta\gamma}\mathbf{h}_\gamma \quad \rightsquigarrow \quad \mathbf{f}_{\alpha\beta} = \sum_{\gamma=1}^{s} f_{\alpha\beta\gamma}\mathbf{e}_\gamma$$

induces an associated syzygy

$$\mathbf{S}_{\alpha\beta} = \frac{\mathbf{t}_{\alpha\beta}}{\mathbf{t}_\alpha}\mathbf{e}_\alpha - \frac{\mathbf{t}_{\alpha\beta}}{\mathbf{t}_\beta}\mathbf{e}_\beta - \mathbf{f}_{\alpha\beta}$$

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Theorem: (Schreyer) $\mathcal{H}$ Gröbner basis for $\prec$ $\Longrightarrow$ $\mathcal{H}_{\text{Schreyer}} = \{\mathbf{S}_{\alpha\beta} \mid 1 \leq \alpha < \beta \leq s\}$ *Gröbner basis* of $\operatorname{Syz}(\mathcal{H})$ for $\prec_{\mathcal{H}}$

- assume $\mathcal{H}$ *involutive basis* for involutive division L
- choose $\mathbf{h}_\alpha \in \mathcal{H}$ and $x_k \in \bar{X}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha)$
- involutive standard representation of non-multiplicative product

$$x_k \mathbf{h}_\alpha = \sum_{\gamma=1}^{s} P_\gamma^{\alpha;k} \mathbf{h}_\gamma$$

induces an associated syzygy

$$\mathbf{S}_{\alpha;k} = x_k \mathbf{e}_\alpha - \sum_{\gamma=1}^{s} P_\gamma^{\alpha;k} \mathbf{e}_\gamma$$

- collect all these syzygies in the set

$$\mathcal{H}_{\text{Syz}} = \left\{ \mathbf{S}_{\alpha;k} \mid 1 \leq \alpha \leq s,\ x_k \in \bar{X}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha) \right\}$$

Lemma: $\mathcal{H}$ involutive basis, $\mathbf{S} = \sum_{\beta=1}^{s} S_\beta \mathbf{e}_\beta \in \mathrm{Syz}(\mathcal{H})$

$$\forall\, 1 \leq \beta \leq s \;:\; S_\beta \in \Bbbk[X_{L,\mathcal{H},\prec}(\mathbf{h}_\beta)] \quad \Longrightarrow \quad \mathbf{S} = \mathbf{0}$$

Proof:

- $\mathbf{S} \in \mathrm{Syz}(\mathcal{H}) \quad \Longrightarrow \quad \sum_{\beta=1}^{s} S_\beta \mathbf{h}_\beta = \mathbf{0}$
- involutive standard representation of $\mathbf{0} \in \langle \mathcal{H} \rangle$ *unique*

  $\Longrightarrow \quad \forall \beta : S_\beta = 0$

Corollary: $\mathcal{H}$ involutive basis $\Longrightarrow$ $\mathrm{Syz}(\mathcal{H}) = \langle \mathcal{H}_{\text{Syz}} \rangle$

Proof:

- take $\mathbf{0} \neq \mathbf{S} \in \mathrm{Syz}(\mathcal{H})$
- Lemma $\Longrightarrow$ at least one component S_β non-multiplicative for $\mathbf{h}_\beta$
- take maximal (wrt $\prec_\mathcal{H}$) non-multiplicative term $cx^\mu\mathbf{e}_\beta$ and maximal non-multiplicative variable x_j with $\mu_j > 0$
- compute $\mathbf{S}' = \mathbf{S} - c(x^\mu/x_j)\mathbf{S}_{\beta;j}$; if $\mathbf{S}' \neq \mathbf{0}$ iterate
- possible new non-multiplicative terms smaller wrt $\prec_\mathcal{H}$ $\Longrightarrow$ iteration terminates with $\mathbf{0}$

Lemma: $\mathcal{H}_{\mathrm{Syz}} \subseteq \mathcal{H}_{\mathrm{Schreyer}}$

Proof: in involutive standard representation

$$x_k \mathbf{h}_\alpha = \sum_{\gamma=1}^{s} P_\gamma^{\alpha;k} \mathbf{h}_\gamma$$

there exists *unique* value β such that $\operatorname{lt}(x_k \mathbf{h}_\alpha) = \operatorname{lt}(P_\beta^{\alpha;k} \mathbf{h}_\beta)$

$\Longrightarrow \quad \mathbf{S}_{\alpha;k} = \mathbf{S}_{\alpha\beta}$

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$\Longrightarrow \quad \mathbf{S}_{\alpha;k} = \mathbf{S}_{\alpha\beta}$

Theorem: $\mathcal{H}$ *involutive basis* for $\prec$ $\Longrightarrow$ $\mathcal{H}_{\mathsf{Syz}}$ *Gröbner basis* of $\operatorname{Syz}(\mathcal{H})$ for $\prec_{\mathcal{H}}$

Proof: corollary to Buchberger's second criterion

Involutive Schreyer Theorem

Goal: (automatic) *involutive basis* of $\operatorname{Syz}(\mathcal{H})$ given an involutive basis $\mathcal{H}$

Solution: currently known *only* for Janet and Pommaret bases

Problems:

- control of leading terms of syzygies $\mathbf{S}_{\alpha;k}$
- "good" numbering of members of $\mathcal{H}$
  (recall: $\prec_{\mathcal{H}}$ depends on numbering)
- control of multiplicative variables assigned to $\mathbf{S}_{\alpha;k}$ by used division

Def: involutive basis $\mathcal{H}$ for division L $\rightsquigarrow$ *L-graph* of $\mathcal{H}$
directed graph with elements $\mathbf{h}$ of $\mathcal{H}$ as vertices
edge from $\mathbf{h}$ to $\mathbf{h}'$, if $\operatorname{lt}\mathbf{h}'$ (unique) involutive divisor of $\operatorname{lt}(x\mathbf{h})$ for some non-multiplicative variable $x \in \bar{X}_{\mathcal{H},L,\prec}(\mathbf{h})$

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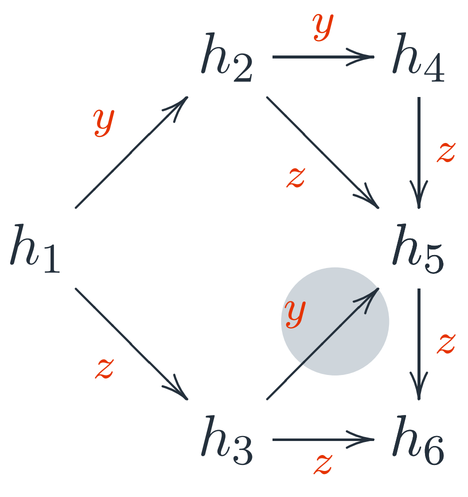
Example: $\mathcal{P} = \Bbbk[x,y,z]$, Pommaret basis $\mathcal{H}$ for $\prec_{\text{degrevlex}}$

$$\mathcal{H} = \{\, h_1 = x^2,\ h_2 = xy,\ h_3 = xz - y,\ h_4 = y^2,\ h_5 = yz - y,\ h_6 = z^2 - z + x \,\}$$

associated P-graph is *acyclic*



Def: involutive basis $\mathcal{H}$ for division L $\rightsquigarrow$ L-*graph* of $\mathcal{H}$
directed graph with elements $\mathbf{h}$ of $\mathcal{H}$ as vertices
edge from $\mathbf{h}$ to $\mathbf{h}'$, if $\operatorname{lt}\mathbf{h}'$ (unique) involutive divisor of $\operatorname{lt}(x\mathbf{h})$ for some non-multiplicative variable $x \in \bar{X}_{\mathcal{H},L,\prec}(\mathbf{h})$

Lemma: L continuous division $\Longrightarrow$ any L-graph acyclic

Proof: cycle corresponds to sequence violating definition of continuity

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Def: L*-ordering* of $\mathcal{H}$ $\rightsquigarrow$ numbering $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\}$
such that $\alpha < \beta$ whenever L-graph contains path from $\mathbf{h}_\alpha$ to $\mathbf{h}_\beta$

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- by lemma above L-ordering always exists for continuous divisions
- in example above $\mathcal{H}$ P-ordered

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such that $\alpha < \beta$ whenever L-graph contains path from $\mathbf{h}_\alpha$ to $\mathbf{h}_\beta$

Lemma: $\mathcal{H}$ L-ordered involutive basis $\Longrightarrow$ $\operatorname{lt}_{\prec_\mathcal{H}} \mathbf{S}_{\alpha;k} = x_k \mathbf{e}_\alpha$

Proof: apply definitions

Def: involutive division L *of Schreyer type* $\rightsquigarrow$
for any involutive basis $\mathcal{H}$ all sets $\bar{X}_{L,\mathcal{H},\prec}(h)$ are again involutive

Involutive Schreyer Theorem

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for any involutive basis $\mathcal{H}$ all sets $\bar{X}_{L,\mathcal{H},\prec}(h)$ are again involutive

Example: Thomas division *not* of Schreyer type

Lemma: Janet and Pommaret division of Schreyer type

Def: involutive division L *of Schreyer type* $\rightsquigarrow$
for any involutive basis $\mathcal{H}$ all sets $\bar{X}_{L,\mathcal{H},\prec}(h)$ are again involutive

Theorem: L continuous involutive division of Schreyer type,
$\mathcal{H}$ L-ordered involutive basis for L and term order $\prec$ $\Longrightarrow$
$\mathcal{H}_{\text{Syz}}$ *involutive basis* of $\operatorname{Syz}(\mathcal{H})$ for L and $\prec_{\mathcal{H}}$

Proof: simple corollary to previous results

- $\mathcal{H}_{\text{Syz}}$ Gröbner basis
- leading terms $x_k\mathbf{e}_\alpha$ with $x_k \in \bar{X}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha)$ because of L-ordering
- $\{x_k\mathbf{e}_\alpha \mid x_k \in \bar{X}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha)\}$ involutive, since L of Schreyer type

Idea: iterate last theorem in order to obtain *free resolution* of polynomial submodule $\mathcal{M} \subseteq \mathcal{P}^m$

Remark: doing this *effectively* requires new computations $\rightsquigarrow$ do not know $(\mathcal{H}_{\text{Syz}})_{\text{Syz}}$, as involutive basis $\mathcal{H}_{\text{Syz}}$ was "for free" (general problem with practical application of Schreyer theorem)

Observation: for *Pommaret division* many statements about obtained resolution possible *without* further computations $\rightsquigarrow$ stronger form of *Hilbert's syzygy theorem*

Theorem: $\mathcal{H}$ Pommaret basis of $\mathcal{M}$, $d = \operatorname{cls}\mathcal{H}$, $\beta_0^{(k)}$ number of generators in $\mathcal{H}$ of class k $\Longrightarrow$ $\mathcal{M}$ has free resolution of the form

$$0 \longrightarrow \mathcal{P}^{r_{n-d}} \longrightarrow \cdots \longrightarrow \mathcal{P}^{r_1} \longrightarrow \mathcal{P}^{r_0} \longrightarrow \mathcal{M} \longrightarrow 0$$

of length $n - d$ with ranks

$$r_i = \sum_{k=1}^{n-i} \binom{n-k}{i} \beta_0^{(k)}$$

(note: r_i upper bound for *Betti number* b_i)

Proof:

- $\operatorname{lt} \mathbf{S}_{\alpha;k} = x_k \mathbf{e}_\alpha \quad \Longrightarrow \quad \operatorname{cls} \mathbf{S}_{\alpha;k} = k \geq \operatorname{cls} \mathbf{h}_\alpha + 1$
  hence $\operatorname{cls} \mathcal{H}_{\text{Syz}} = \operatorname{cls} \mathcal{H} + 1 \quad \rightsquigarrow \quad$ *length* of resolution
  $(\operatorname{cls} \mathcal{H} = n \quad \Longrightarrow \quad \langle \mathcal{H} \rangle$ free module)

Proof:

- $\operatorname{lt} \mathbf{S}_{\alpha;k} = x_k \mathbf{e}_\alpha \implies \operatorname{cls} \mathbf{S}_{\alpha;k} = k \geq \operatorname{cls} \mathbf{h}_\alpha + 1$
  hence $\operatorname{cls} \mathcal{H}_{\text{Syz}} = \operatorname{cls} \mathcal{H} + 1 \rightsquigarrow$ *length* of resolution
  $(\operatorname{cls} \mathcal{H} = n \implies \langle \mathcal{H} \rangle$ free module)
- rank formula obtained by induction and an identity for binomial coefficients
  - $\beta_i^{(k)}$ number of generators of class k in Pommaret basis of $\operatorname{Syz}_i(\mathcal{H})$

$$\implies \qquad r_i = \sum_{k=1}^{n} \beta_i^{(k)}$$

  - definition of Pommaret division

$$\implies \qquad \beta_i^{(k)} = \sum_{j=1}^{k-1} \beta_{i-1}^{(j)}$$

Example: $\mathcal{P} = \Bbbk[x,y,z]$, Pommaret basis $\mathcal{H}$ for $\prec_{\text{degrevlex}}$

$$\mathcal{H} = \{\, h_1 = x^2,\ h_2 = xy,\ h_3 = xz - y,\ h_4 = y^2,\ h_5 = yz - y,\ h_6 = z^2 - z + x \,\}$$

First syzygies (Pommaret basis $\mathcal{H}_{\text{Syz}}$ of $\mathrm{Syz}_1(\mathcal{H})$ for $\prec_{\mathcal{H}}$)

$$\begin{aligned}
\mathbf{S}_{1;3} &= z\mathbf{e}_1 - x\mathbf{e}_3 - \mathbf{e}_2\\
\mathbf{S}_{2;3} &= z\mathbf{e}_2 - x\mathbf{e}_5 - \mathbf{e}_2\\
\mathbf{S}_{3;3} &= z\mathbf{e}_3 - x\mathbf{e}_6 + \mathbf{e}_5 - \mathbf{e}_3 + \mathbf{e}_1\\
\mathbf{S}_{4;3} &= z\mathbf{e}_4 - y\mathbf{e}_5 - \mathbf{e}_4\\
\mathbf{S}_{5;3} &= z\mathbf{e}_5 - y\mathbf{e}_6 + \mathbf{e}_2\\
\mathbf{S}_{1;2} &= y\mathbf{e}_1 - x\mathbf{e}_2\\
\mathbf{S}_{2;2} &= y\mathbf{e}_2 - x\mathbf{e}_4\\
\mathbf{S}_{3;2} &= y\mathbf{e}_3 - x\mathbf{e}_5 + \mathbf{e}_4 - \mathbf{e}_2
\end{aligned}$$

Example: $\mathcal{P} = \Bbbk[x, y, z]$, Pommaret basis $\mathcal{H}$ for $\prec_{\text{degrevlex}}$

$$\mathcal{H} = \{\, h_1 = x^2,\ h_2 = xy,\ h_3 = xz - y,\ h_4 = y^2,\ h_5 = yz - y,\ h_6 = z^2 - z + x \,\}$$

Second syzygies (Pommaret basis $(\mathcal{H}_{\text{Syz}})_{\text{Syz}}$ of $\text{Syz}_2(\mathcal{H})$ for $\prec_{\mathcal{H}_{\text{Syz}}}$)

$$\mathbf{S}_{1;2,3} = z\mathbf{e}_{1;2} - y\mathbf{e}_{1;3} + x\mathbf{e}_{2;3} - x\mathbf{e}_{4;2} - \mathbf{e}_{2;2}$$
$$\mathbf{S}_{2;2,3} = z\mathbf{e}_{2;2} - y\mathbf{e}_{2;3} + x\mathbf{e}_{4;3} - \mathbf{e}_{2;2}$$
$$\mathbf{S}_{3;2,3} = z\mathbf{e}_{3;2} - y\mathbf{e}_{3;3} + x\mathbf{e}_{5;3} + \mathbf{e}_{2;3} - \mathbf{e}_{4;3} - \mathbf{e}_{3;2} + \mathbf{e}_{1;2}$$

all generators of class $3 \quad \Longrightarrow \quad \text{Syz}_2(\mathcal{H})$ free module

Example: $\mathcal{P} = \Bbbk[x, y, z]$, Pommaret basis $\mathcal{H}$ for $\prec_{\text{degrevlex}}$

$$\mathcal{H} = \{ h_1 = x^2,\ h_2 = xy,\ h_3 = xz - y,\ h_4 = y^2,\ h_5 = yz - y,\ h_6 = z^2 - z + x \}$$

free resolution of $\mathcal{I} = \langle \mathcal{H} \rangle$

$$0 \longrightarrow \mathcal{P}^3 \longrightarrow \mathcal{P}^8 \longrightarrow \mathcal{I} \longrightarrow 0$$

or (preferably) of $\mathcal{A} = \mathcal{P}/\mathcal{I}$

$$0 \longrightarrow \mathcal{P}^3 \longrightarrow \mathcal{P}^8 \longrightarrow \mathcal{P}^1 \longrightarrow \mathcal{A} \longrightarrow 0$$

Free Resolutions of Monomial Modules

Assume $\mathcal{M} \subset \mathcal{P}^m$ quasi-stable *monomial* module with Pommaret basis $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_p\}$ $\Longrightarrow$ *explicit* presentation of resolution exists (not requiring any further computations!)

Assume $\mathcal{M} \subset \mathcal{P}^m$ quasi-stable *monomial* module with Pommaret basis $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_p\} \Longrightarrow$ *explicit* presentation of resolution exists (not requiring any further computations!)

Explicit expressions for all syzygies obtainable from P*-graph*

- let x_k be non-multiplicative for generator $\mathbf{h}_\alpha$
- $\mathcal{H}$ contains generator $\mathbf{h}_\beta$ with $x_k\mathbf{h}_\alpha = x^\mu \mathbf{h}_\beta$ and $x^\mu \in \mathbb{k}[X_P(\mathbf{h}_\beta)]$
- write $\Delta(\alpha, k) = \beta$ and $t_{\alpha,k} = x^\mu$

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- write $\Delta(\alpha, k) = \beta$ and $t_{\alpha,k} = x^\mu$

Theorem: Let $\mathbf{k} = (k_1, \dots, k_i)$ with $\operatorname{cls} \mathbf{h}_\alpha < k_1 < \cdots < k_i$

$$\mathbf{S}_{\alpha;\mathbf{k}} = \sum_{j=1}^{i} (-1)^{i-j} \left(x_{k_j} \mathbf{S}_{\alpha;\mathbf{k}_j} - t_{\alpha,k_j} \mathbf{S}_{\Delta(\alpha,k_j);\mathbf{k}_j} \right)$$

where $\mathbf{k}_j = (k_1, \dots, \widehat{k_j}, \dots, k_i)$ ($\mathbf{k}$ with jth entry removed)

Remark: For quasi-stable ideals the resolution can always be given the structure of a *differential algebra*.

- let h_α, h_β be two elements of $\mathcal{H}$
- $\mathcal{H}$ contains generator $\mathbf{h}_\gamma$ with $\mathbf{h}_\alpha \mathbf{h}_\beta = x^\mu \mathbf{h}_\gamma$ and $x^\mu \in \Bbbk[X_P(\mathbf{h}_\gamma)]$
- write $\Gamma(\alpha, \beta) = \gamma$ and $m_{\alpha,\beta} = x^\mu$
- express resolution as complex with symmetric and anti-symmetric part; use m, Γ to define product on symmetric part; use exterior product on anti-symmetric part
- properties of Pommaret basis ensure associativity and Leibniz rule

(Same construction possible for polynomial case; however, obtained product in general not associative and does not satisfy Leibniz rule.)

Free resolution of *graded* module $\mathcal{M}$ *minimal* $\rightsquigarrow$
all maps $\phi_i : \mathcal{P}^{r_i} \longrightarrow \mathcal{P}^{r_{i-1}}$ in the resolution

- described by matrices with all entries of *positive* degrees (i. e. without constant terms) or equivalently
- map standard basis to *minimal* generating set of image

Theorem: Minimal free resolution unique up to isomorphism.

Remark: any non-minimal resolution can be transformed into a minimal one with some linear algebra.

Def: *projective dimension* $\operatorname{pdim} \mathcal{M}$ $\rightsquigarrow$ length of minimal free resolution

Lemma: Resolution obtained with Pommaret basis minimal $\Longleftrightarrow$ all syzygies $\mathbf{S}_{\alpha;k} \in \mathcal{H}_{\mathrm{Syz}}$ free of constant terms

Proof: follows easily from analysis of $\mathbf{S}_{\alpha;k_1,k_2}$

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Thus in general Pommaret basis does *not* yield minimal resolution. However, much information about minimal resolution deducible!

Theorem: $\mathcal{H}$ Pommaret basis of $\mathcal{M}$ for class respecting term order and

$$\operatorname{cls}\mathcal{H} = d \quad \Longrightarrow \quad \operatorname{pdim}\mathcal{M} = n - d$$

Proof: analyse minimisation process applied to resolution obtained from $\mathcal{H}$ $\rightsquigarrow$ minimisation cannot reduce length of resolution (analyse syzygies obtained from generator of minimal class and maximal degree)

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Corollary: (Auslander-Buchsbaum formula)

$$\operatorname{depth} \mathcal{M} + \operatorname{pdim} \mathcal{M} = n$$

Thm: $\mathcal{M}$ monomial module with Pommaret basis $\mathcal{H}$

$\mathcal{M}$ *stable* $\Longleftrightarrow$ $\mathcal{H}$ minimal basis of $\mathcal{M}$ $\Longleftrightarrow$ resolution obtained from $\mathcal{H}$ minimal *(Eliahou-Kervaire resolution)*

Thm: $\mathcal{M}$ monomial module with Pommaret basis $\mathcal{H}$
$\mathcal{M}$ *stable* $\Longleftrightarrow$ $\mathcal{H}$ minimal basis of $\mathcal{M}$ $\Longleftrightarrow$
resolution obtained from $\mathcal{H}$ minimal *(Eliahou-Kervaire resolution)*

Thm: $\mathcal{M}$ polynomial module with Pommaret basis $\mathcal{H}$
resolution obtained from $\mathcal{H}$ minimal $\Longrightarrow$ $\mathcal{M}$ *componentwise linear*

(for “proper” – generic – choice of δ-regular variables, converse true, too)

Def: graded module $\mathcal{M}$ *q-regular* $\rightsquigarrow$

- $\mathcal{M}$ can be generated in degree $\leq q$
- $\mathrm{Syz}_j(\mathcal{M})$ can be generated in degree $\leq q + j$

Castelnuovo-Mumford regularity of $\mathcal{M}$ $\rightsquigarrow$

$$\operatorname{reg} \mathcal{M} = \min \{ q \in \mathbb{N} \mid \mathcal{M}\ q\text{-regular} \}$$

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- $\operatorname{Syz}_j(\mathcal{M})$ can be generated in degree $\leq q + j$

Castelnuovo-Mumford regularity of $\mathcal{M}$ $\rightsquigarrow$
$\operatorname{reg}\mathcal{M} = \min\{q \in \mathbb{N} \mid \mathcal{M}\ q\text{-regular}\}$

$\operatorname{reg}\mathcal{M}$ crucial for *complexity* analysis of Gröbner bases:

Theorem: (Bayer-Stillman)
in generic variables $\deg\mathcal{G} \geq \operatorname{reg}\mathcal{M}$ for any Gröbner basis $\mathcal{G}$
(generically equality for `degrevlex`)

Problem: what means *generic*? No *effective* test known. . .

Theorem: $\mathcal{H}$ Pommaret basis of $\mathcal{M}$ for `degrevlex` $\iff$ $\deg \mathcal{H} = \operatorname{reg} \mathcal{M}$

Proof:

- "$\geq$" obvious from resolution induced by $\mathcal{H}$
- for "$=$" take element $\mathbf{h}_\alpha \in \mathcal{H}$ of *maximal degree* $\deg \mathcal{H}$ and of *minimal class* d among all elements of degree $\deg \mathcal{H}$ $\rightsquigarrow$ show that syzygy $\mathbf{S}_{\alpha;d+1,d+2,\dots,n}$ cannot be eliminated during minimisation process

Theorem: $\mathcal{H}$ Pommaret basis of $\mathcal{M}$ for `degrevlex` $\Longleftrightarrow$ $\deg \mathcal{H} = \operatorname{reg} \mathcal{M}$

Proof:

- "$\geq$" obvious from resolution induced by $\mathcal{H}$
- for "$=$" take element $\mathbf{h}_\alpha \in \mathcal{H}$ of *maximal degree* $\deg \mathcal{H}$ and of *minimal class* d among all elements of degree $\deg \mathcal{H}$ $\leadsto$ show that syzygy $\mathbf{S}_{\alpha;d+1,d+2,\dots,n}$ cannot be eliminated during minimisation process

Remark: iteration of this argument $\leadsto$ all *extremal Betti numbers* of $\mathcal{M}$ can be read off `degrevlex` Pommaret basis $\mathcal{H}$

Example: recall from first lecture

$$\mathcal{I} = \langle z^8 - wxy^6,\ y^7 - x^6z,\ yz^7 - wx^7 \rangle \lhd \mathbb{k}[w, x, y, z]$$

(reduced) Gröbner basis for `degrevlex`

chosen variables already δ-regular
completion adds the polynomials $z^k(y^7 - x^6z)$ for $1 \leq k \leq 6$ $\rightsquigarrow$
Pommaret basis $\mathcal{H}$ with $\deg \mathcal{H} = 13$ $\Longrightarrow$

$$\operatorname{reg} \mathcal{I} = 13$$

Some classical results on $\operatorname{reg}\mathcal{M}$ can be obtained as easy corollaries.

Theorem: (Eisenbud-Goto)
$\mathcal{M}$ q-regular $\iff$ truncation $\mathcal{M}_{\geq q}$ possesses *linear* free resolution

Proof: "$\Longrightarrow$": consider `degrevlex` Pommaret basis $\mathcal{H}$ $\rightsquigarrow$
$\deg\mathcal{H} = \operatorname{reg}\mathcal{M} \leq q$ $\rightsquigarrow$ $\mathcal{H}_q$ Pommaret basis of $\mathcal{M}_{\geq q}$ with all generators of same degree $\rightsquigarrow$ induced resolution minimal and linear

"$\Longleftarrow$": $\mathcal{M}_{\geq q}$ has linear resolution $\rightsquigarrow$ $\operatorname{reg}\mathcal{M}_{\geq q} = q$ $\rightsquigarrow$ $\mathcal{M}_{\geq q}$ has Pommaret basis of degree q $\rightsquigarrow$ $\mathcal{M}$ has Pommaret basis $\mathcal{H}$ with $\operatorname{reg}\mathcal{M} = \deg\mathcal{H} \leq q$ $\rightsquigarrow$ $\mathcal{M}$ q-regular

(noted as "curiosité" already 20 years earlier by Serre in the context of differential equations)

Some classical results on $\operatorname{reg}\mathcal{M}$ can be obtained as easy corollaries.

Theorem: (Bayer-Stillman)
homogeneous ideal $\mathcal{I} \subseteq \mathcal{P}$ q-regular $\iff$ $\exists\, y_1, \dots, y_d \in \mathcal{P}_1$

$$\bigl(\langle \mathcal{I}, y_1, \dots, y_{k-1}\rangle : y_k\bigr)_q = \langle \mathcal{I}, y_1, \dots, y_{k-1}\rangle_q$$

$$\langle \mathcal{I}, y_1, \dots, y_d\rangle_q = \mathcal{P}_q$$

"Proof:" $y_1, \dots, y_d$ can be extended to δ-regular variables

One computation (Pommaret basis for `degrevlex` *plus* δ-regular coordinates) yields all the following information:

- Gröbner basis
- (complementary) Rees decomposition
- Hilbert series (function, polynomial)
- Krull dimension
  (with maximal set of independent variables)
- multiplicity
- depth
  (with simple maximal regular sequence)
- test for Cohen-Macaulay module
- test for Gorenstein module
  (with socle basis)

One computation (Pommaret basis for `degrevlex` *plus* δ-regular coordinates) yields all the following information:

- projective dimension
  (plus bounds on all Betti numbers)
- Castelnuovo-Mumford regularity
  (plus all extremal Betti numbers)
- Noether normalisation
- Saturation $\mathcal{I}^{\mathrm{sat}}$
- parameter ideal
- test for componentwise linearity
- ... *work in progress* ...