Involutive Bases IV

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- **General Involutive Bases**
- **Basic Algorithms**
- **Pommaret Bases and δ-Regularity**
- **Combinatorial Decompositions and Applications**
  - Stanley decompositions of $\mathcal{I}$ and $\mathcal{P}/\mathcal{I}$
  - Hilbert functions and polynomials
  - Dimension and Depth
  - Cohen-Macaulay and Gorenstein rings
- **Syzygy Theory and Applications**

Recall: underlying idea of involutive bases $\rightsquigarrow$ write homogeneous ideal $\mathcal{I} \lhd \Bbbk[X]$ as *direct* sum of polynomial rings $\Bbbk[X' \subseteq X]$

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$$\mathcal{M} \cong \bigoplus_{t \in \mathcal{T}} \mathbb{k}[X_t] \cdot t$$

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- *Rees decomposition:* $\forall t \in \mathcal{T} : \mathcal{X}_t = \{x_1, \dots, x_{\operatorname{lev} t}\}$ (with *level* $\operatorname{lev} t$ of generator t)
- *quasi-Rees decomposition:* $\exists \bar{t} \in \mathcal{T}\ \forall t \in \mathcal{T} : \mathcal{X}_t \subseteq \mathcal{X}_{\bar{t}}$

$\mathcal{H}$ *any* involutive basis of $\mathcal{I} \lhd \mathcal{P}$ $\rightsquigarrow$ Stanley decomposition of $\mathcal{I}$
($\mathcal{H}$ Pommaret basis $\rightsquigarrow$ Rees decomposition with $\operatorname{lev} t = \operatorname{cls} t$)

Problem: Stanley decomposition of $\mathcal{A} = \mathcal{P}/\mathcal{I}$

Arbitrary involutive basis $\mathcal{H}$ of $\mathcal{I}$ does generally *not* yield such a *complementary decomposition*!

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Prop: every ideal $\mathcal{I} \lhd \mathcal{P}$ has a complementary decomposition

Proof:

- Macaulay: $\mathcal{A} \cong \mathcal{A}' = \mathcal{P}/\operatorname{lt}\mathcal{I}$ as $\Bbbk$-linear spaces $\rightsquigarrow$ *monomial* (i. e. combinatorial) problem
- induction over number of variables X yields simple recursive *algorithm* for construction of complementary decomposition via “slicing”
- alternative algorithm via *Janet bases*

Prop: $\mathcal{H}$ *Pommaret basis* of *monomial* ideal $\mathcal{I} \lhd \mathcal{P}$; define
$\mathcal{B}_0 = \{t \in \mathbb{T}(X) \setminus \mathcal{I} \mid \deg t < \deg \mathcal{H}\}$ and
$\mathcal{B}_1 = \{t \in \mathbb{T}(X) \setminus \mathcal{I} \mid \deg t = \deg \mathcal{H}\}$ $\Longrightarrow$
complementary *Rees decomposition*

$$\mathcal{A} \cong \langle \mathcal{B}_0 \rangle_{\mathbb{k}} \oplus \bigoplus_{t \in \mathcal{B}_1} \mathbb{k}[x_1, \dots, x_{\operatorname{cls} t}] \cdot t$$

(i. e. $\operatorname{lev} t = 0$ for $t \in \mathcal{B}_0$ and $\operatorname{lev} t = \operatorname{cls} t$ for $t \in \mathcal{B}_1$)

Proof: $\mathcal{P}_{\geq q} = \bigoplus_{\deg t = q} \mathbb{k}[x_1, \dots, x_{\operatorname{cls} t}] \cdot t$

(above Rees decomposition usually highly redundant; optimised form with less generators obtainable with algorithm of *Hironaka*)

Example: $\mathcal{I} = \langle z^3,\ yz^2 - xz^2,\ y^2 - xy\rangle \lhd \mathbb{k}[x,y,z],\quad \prec = \prec_{\text{degrevlex}}$
Pommaret basis: $\mathcal{H} = \{z^3,\ yz^2 - xz^2,\ y^2z - xyz,\ y^2 - xy\}$

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complementary Rees decomposition according to proposition

$$\mathcal{A} \cong \langle 1, x, y, z, x^2, xy, xz, yz, z^2\rangle_{\mathbb{k}} \oplus \langle x^3, x^2y, x^2z, xyz, xz^2\rangle_{\mathbb{k}[x]}$$

complementary Rees decomposition with Hironaka algorithm

$$\mathcal{A} \cong \langle 1, y, z, yz, z^2\rangle_{\mathbb{k}[x]}$$

(note: all generators possess the *same level* $\rightsquigarrow$ see end of lecture) consider *monomial* ideal $\mathcal{I} \lhd \mathcal{P}$

Def: *Stanley depth* of $\mathcal{I}$ $\rightsquigarrow$

$$\operatorname{S-depth}\mathcal{I} = \max\left\{k \in \mathbb{N} \mid \exists \text{ Stanley decomp. with } \min_{t\in\mathcal{T}} |X_t| = k\right\}$$

Conjecture: $\operatorname{S-depth}\mathcal{I} \geq \operatorname{depth}\mathcal{I}$

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Prop: Stanley conjecture true for *quasi-stable* ideals and their quotients

Proof: will see later that Pommaret basis induces Rees decomposition of $\mathcal{I}$ with $\min_{t\in\mathcal{T}} |X_t| = \text{depth}\,\mathcal{I}$ and Rees decomposition of $\mathcal{A}$ with $\min_{t\in\mathcal{T}} |X_t| = \text{depth}\,\mathcal{A} = \text{depth}\,\mathcal{I} - 1$

Recall: $\mathcal{M}$ non-negatively graded $\mathcal{P}$-module $\rightsquigarrow$

- *Hilbert function:* $h_{\mathcal{M}}(q) = \dim_{\Bbbk} \mathcal{M}_q$
- *Hilbert series:* $\mathcal{H}_{\mathcal{M}}(\lambda) = \sum_{q \geq 0} h_{\mathcal{M}}(q)\lambda^q$

  (generating function of $h_{\mathcal{M}}$)

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Prop: $\mathcal{P} = \Bbbk[x_1, \dots, x_n] \implies$

$$\mathcal{H}_{\mathcal{M}}(\lambda) = \frac{f(\lambda)}{(1-\lambda)^n} \quad \text{with } f \in \mathbb{Z}[\lambda]$$

Cancelling common factors yields $\mathcal{H}_{\mathcal{M}}(\lambda) = g(\lambda)/(1-\lambda)^D$ where $D = \dim \mathcal{M}$

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Thm: there exists *Hilbert polynomial* $H_{\mathcal{M}} \in \mathbb{Q}[s]$ such that $\forall q \geq \deg g : h_{\mathcal{M}}(q) = H_{\mathcal{M}}(q)$

Prop: $\mathcal{M}$ has Stanley decomposition $\mathcal{M} \cong \bigoplus_{t\in\mathcal{T}} \mathbb{k}[X_t] \cdot t \quad \Longrightarrow$

Hilbert series $$\mathcal{H}_{\mathcal{M}}(\lambda) = \sum_{t\in\mathcal{T}} \frac{\lambda^{q_t}}{(1-\lambda)^{k_t}}$$

Hilbert function $$h_{\mathcal{M}}(q) = \sum_{t\in\mathcal{T}} \binom{q - q_t + k_t - 1}{q - q_t}$$

with $q_t = \deg t$ and $k_t = |X_t|$

Proof: $\mathcal{H}_{\mathbb{k}[x_1,\dots,x_k]}(\lambda) = 1/(1-\lambda)^k$

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Proof: $\mathcal{H}_{\mathbb{k}[x_1,\dots,x_k]}(\lambda) = 1/(1-\lambda)^k$

Cor: Stanley decomposition as above $\implies$

$$\dim \mathcal{M} = \max_{t\in\mathcal{T}} |X_t| \qquad \deg \mathcal{M} = \#\{t\in\mathcal{T} : |X_t| = \dim\mathcal{M}\}$$

Prop: $\mathcal{H}$ Pommaret basis of $\mathcal{I}$ with $\deg \mathcal{H} = q \implies$

$$\dim \mathcal{A} = D = \min\{i \mid \langle \mathcal{H}, x_1, \dots, x_i \rangle_q = \mathcal{P}_q\}$$

Proof:

- Hilbert polynomials of $\mathcal{A}$ and $\mathcal{A}_{\geq q}$ coincide
- consider Pommaret basis $\mathcal{H}_q$ of $\mathcal{I}_{\geq q}$
  - *all* terms $t \in \mathbb{T}(X)$ with $\deg t = q$ and $\operatorname{cls} t > D$ contained in $\operatorname{lt} \mathcal{H}_q$ (otherwise we needed also $x_{D+1}, \dots$)
  - there *exists* term $s \in \mathbb{T}(X)$ with $\deg s = q$ and $\operatorname{cls} s = D$ *not* contained in $\operatorname{lt} \mathcal{H}_q$ (otherwise x_D was not needed)
- s generator in complementary Rees decomposition with maximal number of multiplicative variables $\{x_1, \dots, x_D\} \implies \dim \mathcal{A} = D$

Def: $X_{\mathcal{I}} \subseteq X$ *independent modulo* $\mathcal{I}$ $\rightsquigarrow$ $\mathcal{I} \cap \mathbb{k}[\mathcal{X}_{\mathcal{I}}] = 0$
$X_{\mathcal{I}}$ *strongly independent modulo* $\mathcal{I}$ and $\prec$ $\rightsquigarrow$ $\operatorname{lt} \mathcal{I} \cap \mathbb{k}[\mathcal{X}_{\mathcal{I}}] = 0$

Prop: $\dim \mathcal{A} = \max \{ |X_{\mathcal{I}}| : X_{\mathcal{I}}$ (strongly) independent modulo $\mathcal{I} \}$

Problem: in general *many* different maximal (strongly) independent sets (not necessarily of the same size!) $\rightsquigarrow$ *combinatorial explosion* for larger number of variables

Lemma: let $\mathcal{T}$ define *quasi-Rees decomposition* of $\mathcal{A}'$ with maximal set $X_{\bar{t}}$ of multiplicative variables, $\mathcal{B}$ minimal basis of $\operatorname{lt}\mathcal{I}$ $\Longrightarrow$

$$x \notin X_{\bar{t}} \quad \Longleftrightarrow \quad \exists e \in \mathbb{N} : x^e \in \mathcal{B}$$

Proof:

"$\Rightarrow$" $x \notin X_{\bar{t}} \Longrightarrow \forall t \in \mathcal{T} : x \notin X_t$
$\mathcal{T}$ finite set $\Longrightarrow$ $\mathcal{T}$ contains only finitely many terms x^k $\Longrightarrow$
$\operatorname{lt}\mathcal{I}$ and thus minimal basis $\mathcal{B}$ contains term x^e

"$\Leftarrow$" $x^e \in \operatorname{lt}\mathcal{I} \Longrightarrow \forall t \in \mathcal{T} : x^e \cdot t \in \operatorname{lt}\mathcal{I} \Longrightarrow$
$\forall t \in \mathcal{T} : x \notin X_t$

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$$x \notin X_{\bar{t}} \iff \exists\, e \in \mathbb{N} : x^e \in \mathcal{B}$$

Prop: assumptions as above $\Longrightarrow$
$X_{\bar{t}}$ *unique* maximal strongly independent set modulo $\mathcal{I}$

Proof: $s \in \operatorname{lt}\mathcal{I} \cap \Bbbk[X_{\bar{t}}] \Longrightarrow s\bar{t} \in \operatorname{lt}\mathcal{I} \rightsquigarrow$
impossible as $\mathcal{T}$ defines Stanley decomposition of $\mathcal{A}'$ $\Longrightarrow$
$X_{\bar{t}}$ strongly independent set modulo $\mathcal{I}$; rest follows from Lemma

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Cor: variables δ-regular for $\mathcal{I}$ $\Longrightarrow$
$\{x_1, \dots, x_D\}$ unique maximal strongly independent set modulo $\mathcal{I}$

Recall: *Noether normalisation* of $\mathcal{A}$ $\rightsquigarrow$
injective map $\phi : \mathbb{k}[y_1, \dots, y_D] \hookrightarrow \mathcal{A}$ with $\operatorname{im}\phi \subseteq \mathcal{A}$ *integral* ring extension
(in particular, $\mathcal{A}$ *finitely generated* $\mathbb{k}[y_1, \dots, y_D]$-module)

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Prop: let $\mathcal{T}$ define *quasi-Rees decomposition* of $\mathcal{A}'$ with maximal set $X_{\bar{t}}$ of multiplicative variables $\Longrightarrow$ restriction of canonical map $\mathcal{P} \twoheadrightarrow \mathcal{A}$ to $\mathbb{k}[X_{\bar{t}}]$ Noether normalisation

Proof: $X_{\bar{t}}$ strongly independent set modulo $\mathcal{I}$ $\Longrightarrow$ $X_{\bar{t}}$ independent set modulo $\mathcal{I}$ $\Longrightarrow$ ϕ injective

definition of quasi-Rees decomposition $\Longrightarrow$ $\mathcal{A}$ finitely generated $\mathbb{k}[X_{\bar{t}}]$-module

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Prop: let $\mathcal{T}$ define *quasi-Rees decomposition* of $\mathcal{A}'$ with maximal set $X_{\bar{t}}$ of multiplicative variables $\Longrightarrow$
restriction of canonical map $\mathcal{P} \twoheadrightarrow \mathcal{A}$ to $\mathbb{k}[X_{\bar{t}}]$ Noether normalisation

Cor: variables δ-regular for $\mathcal{I}$ $\Longrightarrow$
$\mathbb{k}[x_1, \dots, x_D]$ defines Noether normalisation of $\mathcal{A}$

Some comments:

- existence proof for *Pommaret bases* $\Longrightarrow$
  existence proof for *(quasi-)Rees decomposition* $\Longrightarrow$
  existence proof for *Noether normalisation*
- construction of δ-*regular* variables $\rightsquigarrow$ put $\mathcal{I}$ in *Noether position*
  last lecture: *deterministic* approach possible!
- converse of corollary *not* true: even if $\mathbb{k}[x_1, \dots, x_D]$ defines Noether normalisation of $\mathcal{A}$, variables not necessarily δ-regular

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Prop: $\mathcal{I}$ *monomial* ideal with $D = \dim \mathcal{A}$, $\mathcal{I} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_r$ irredundant monomial primary decomposition with $D_j = \dim(\mathcal{P}/\mathfrak{q}_j)$

$\mathcal{I}$ quasi-stable $\iff$ $\mathbb{k}[x_1, \dots, x_D]$ Noether normalisation of $\mathcal{A}$ and $\mathbb{k}[x_1, \dots, x_{D_j}]$ Noether normalisation of $\mathcal{P}/\mathfrak{q}_j$ for all j

Proof: $\mathcal{I}$ quasi-stable $\iff$ $\mathcal{I}$ of nested type
(see last lecture)

Recall: $\mathcal{M}$ finitely generated (graded) polynomial module, $\mathcal{J} \lhd \mathcal{P}$ proper (homogeneous) ideal

- sequence $(f_1, \dots, f_r)$ of polynomials $f_k \in \mathcal{J}$ is $\mathcal{M}$*-regular* $\rightsquigarrow$
  f_1 non zero divisor on $\mathcal{M}$ and
  f_k non zero divisor on $\mathcal{M}/\langle f_1, \dots, f_{k-1}\rangle\mathcal{M}$
- *depth* of module $\mathcal{M}$ on ideal $\mathcal{J}$ $\rightsquigarrow$
  maximal length $\operatorname{depth}(\mathcal{J}, \mathcal{M})$ of $\mathcal{M}$-regular sequence in $\mathcal{J}$

  - for analysis of module $\mathcal{M}$ mainly $\operatorname{depth}(\mathfrak{m}, \mathcal{M})$ with $\mathfrak{m} = \langle x_1, \dots, x_n\rangle$ (suffices to consider $f_1, \dots, f_r \in \mathcal{P}_1$)
  - for analysis of ideal $\mathcal{J}$ mainly $\operatorname{depth}(\mathcal{J}, \mathcal{P})$ with $\mathcal{P}$ considered as $\mathcal{P}$-module

- for ideal $\mathcal{I} \lhd \mathcal{P}$ and module $\mathcal{A} = \mathcal{P}/\mathcal{I}$ $\rightsquigarrow$ (“abstract nonsense”)

$$\operatorname{depth}(\mathfrak{m}, \mathcal{A}) = \operatorname{depth}(\mathfrak{m}, \mathcal{I}) - 1$$

Computational methods: (for $\operatorname{depth}(\mathfrak{m}, \mathcal{M})$)

- compute (length of) *minimal free resolution* of $\mathcal{M}$ or
- compute *extension groups* $\operatorname{Ext}^{n-i}_{\mathcal{P}}(\mathcal{M}, \Bbbk)$ till non-vanishing group found

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Def: $\mathcal{I} \lhd \mathcal{P}$ ideal of codimension $c = n - \dim \mathcal{I}$
maximal system of parameters for $\mathcal{I}$ $\rightsquigarrow$
$\mathcal{P}$-regular sequence $(f_1, \dots, f_c)$ in $\mathcal{I}$ such that $\operatorname{codim} \langle f_1, \dots, f_c \rangle = c$

Remark: $\tilde{\mathcal{I}} = \langle f_1, \dots, f_c \rangle \subseteq \mathcal{I}$ *parameter ideal* for $\mathcal{I}$ $\rightsquigarrow$
complete intersection used e. g. in algorithms for primary decomposition
(determination of parameter ideals in practice often bottle neck)

Prop: $\mathcal{H}$ Pommaret basis of ideal $\mathcal{I}$ with $\operatorname{codim}\mathcal{I} = c \implies$ $\mathcal{H}$ contains generators $h_1, \dots, h_c$ with $\operatorname{lt} h_i = x_{n-i+1}^{e_i}$ which form maximal system of parameters

Proof:

- generators $h_1, \dots, h_c$ define Gröbner basis of ideal spanned by them (leading terms coprime $\leadsto$ Buchberger's first criterion)
- $\operatorname{Syz}(h_1, \dots, h_c)$ generated by trivial syzygies $h_i\mathbf{e}_j - h_j\mathbf{e}_i$ (Schreyer's theorem on syzygies of Gröbner basis)
- $fh_k \in \langle h_1, \dots, h_{k-1}\rangle \leadsto$ induces syzygy in $\operatorname{Syz}(h_1, \dots, h_k)$ with component $f\mathbf{e}_k \implies f \in \langle h_1, \dots, h_{k-1}\rangle \implies$ h_k non zero divisor on $\mathcal{P}/\langle h_1, \dots, h_{k-1}\rangle$

Theorem: $\mathcal{H}$ Pommaret basis of $\mathcal{I}$ for $\prec_{\text{degrevlex}}$ and $d = \operatorname{cls}\mathcal{H}$
$\Longrightarrow$ $(x_1, \dots, x_d)$ *maximal* $\mathcal{I}$-regular sequence, i. e.

$$\operatorname{depth}(\mathfrak{m}, \mathcal{I}) = d \qquad \text{and} \qquad \operatorname{depth}(\mathfrak{m}, \mathcal{A}) = d - 1$$

Proof:

- $\mathcal{I}$-regularity follows from induced Rees decomposition of $\mathcal{I}$
- not possible to extend sequence with x_k where $k > d$:
  take generator $h \in \mathcal{H}$ with $\operatorname{cls} h = d$ and of maximal degree among all such generators $\rightsquigarrow$ analysis of involutive standard representation of non-multiplicative product $x_k h$ shows that x_k zero divisor on $\mathcal{I}/\langle x_1, \dots, x_d\rangle\mathcal{I}$
- existence of longer $\mathcal{I}$-regular sequence in $\mathcal{P}_1$ $\rightsquigarrow$ contradiction to δ-regularity of variables X

Remark: no similar result known for any other division!

Cor: $\operatorname{depth}(\mathfrak{m}, \mathcal{A}) \leq \dim \mathcal{A}$

Remark: consider truncation $\mathcal{A}_{\geq q}$ with $q = \deg \mathcal{H}$ for Pommaret basis $\mathcal{H}$
$\rightsquigarrow$ Pommaret basis $\mathcal{H}_q$ of $\mathcal{I}_{\geq q}$ $\rightsquigarrow$

everything interesting happens between $\operatorname{depth}(\mathfrak{m}, \mathcal{A})$ *and* $\dim \mathcal{A}$

$\operatorname{lt} \mathcal{I}$ contains *all* terms t of degree q with $\operatorname{cls} t > \dim \mathcal{A}$ and *no* term t of degree q with $\operatorname{cls} t \leq \operatorname{depth}(\mathfrak{m}, \mathcal{A})$

Def: $\mathcal{A} = \mathcal{P}/\mathcal{I}$ *Cohen-Macaulay* $\rightsquigarrow$ $\operatorname{depth}(\mathfrak{m}, \mathcal{A}) = \dim \mathcal{A}$

Problem: *effective* test for affine algebra $\mathcal{A} = \mathcal{P}/\mathcal{I}$ to be Cohen-Macaulay

- *Classically:* difficult part to determine $\operatorname{depth}(\mathfrak{m}, \mathcal{A})$
- trivial with *Pommaret basis* of $\mathcal{I}$ for $\prec_{\text{degrevlex}}$

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- trivial with *Pommaret basis* of $\mathcal{I}$ for $\prec_{\text{degrevlex}}$

Prop: if variables X δ-regular for $\mathcal{I}$ and $\prec_{\text{degrevlex}}$, then

$$\mathcal{P}/\mathcal{I} \text{ Cohen-Macaulay} \iff \mathcal{P}/\operatorname{lt}\mathcal{I} \text{ Cohen-Macaulay}$$

(*not* true without made assumptions!)

Def: $\mathcal{A} = \mathcal{P}/\mathcal{I}$ *Cohen-Macaulay* $\rightsquigarrow$ $\operatorname{depth}(\mathfrak{m}, \mathcal{A}) = \dim \mathcal{A}$

Cor: *(Hironaka criterion)* $\mathcal{A}$ Cohen-Macaulay ring $\Longleftrightarrow$
$\mathcal{A}$ has Rees decomposition where all generators have the same level

Proof:

"$\Leftarrow$": all generators level d $\Longrightarrow$ $\operatorname{depth}(\mathfrak{m}, \mathcal{A}) = \dim \mathcal{A} = d$

"$\Rightarrow$": (after variable transformation) $\mathcal{I}$ has Pommaret basis $\mathcal{H}$ with
$\operatorname{cls} \mathcal{H} = d + 1$ $\rightsquigarrow$ consider $\mathcal{B} = \{x^\nu \in \overline{\langle \operatorname{lt} \mathcal{H} \rangle} \mid \operatorname{cls} \nu > d\}$ $\rightsquigarrow$
$\mathcal{B}$ finite set by characterisation of $\dim \mathcal{A} = d$ $(|\nu| < \deg \mathcal{H})$ $\rightsquigarrow$
Rees decomposition $\mathcal{A} \cong \bigoplus_{x^\nu \in \mathcal{B}} \Bbbk[x_1, \dots, x_d] \cdot x^\nu$
(obtainable by applying Janet's algorithm to Janet basis $\mathcal{H}$)

Example: recall from above

$$\mathbb{k}[x,y,z]/\langle z^3,\, yz^2 - xz^2,\, y^2 - xy\rangle \cong \langle 1, y, z, yz, z^2\rangle_{\mathbb{k}[x]}$$

Hironaka criterion $\Longrightarrow$ Cohen-Macaulay ring

- $\langle \mathcal{H}, x\rangle_{\geq 3} = \mathcal{P}_{\geq 3} \quad \Longrightarrow \quad \dim \mathcal{A} = 1$
- $\operatorname{cls} \mathcal{H} = 2 \quad \Longrightarrow \quad \operatorname{depth}(\mathfrak{m}, \mathcal{A}) = 1$

Recall: *socle* of $\mathcal{P}$-module $\mathcal{M}$ $\rightsquigarrow$

$$\operatorname{Soc}\mathcal{M} = 0 :_{\mathcal{M}} \mathfrak{m} = \{m \in \mathcal{M} \mid \mathfrak{m} \cdot m = 0\}$$

consider d-dimensional affine algebra $\mathcal{A} = \mathcal{P}/\mathcal{I}$ with system of parameters $a_1, \dots, a_d \in \mathcal{A}$ (i. e. $\dim \bar{\mathcal{A}} = 0$ for $\bar{\mathcal{A}} = \mathcal{A}/\langle a_1, \dots, a_d\rangle$)

Def: assume $\mathcal{A}$ Cohen-Macaulay

- *type* of $\mathcal{A}$ $\rightsquigarrow$ $t = \dim_{\mathbb{k}} \operatorname{Soc}\bar{\mathcal{A}}$ (value of t independent of chosen system of parameters)
- $\mathcal{A}$ *Gorenstein* $\rightsquigarrow$ $t = 1$

Problem: *effective* test for affine Cohen-Macaulay algebra $\mathcal{A} = \mathcal{P}/\mathcal{I}$ to be Gorenstein

$\mathcal{H}$ Pommaret basis of $\mathcal{I}$ for $\prec_{\text{degrevlex}}$

- $\operatorname{depth}(\mathfrak{m}, \mathcal{A}) = d \implies \operatorname{cls}\mathcal{H} = d + 1$
- $\dim \mathcal{A} = d \implies$
  $a_1 = x_1 + \mathcal{I}, \dots, a_d = x_d + \mathcal{I}$ system of parameters
- $\bar{\mathcal{A}} = \mathcal{A}/\langle a_1, \dots, a_d \rangle \cong \mathcal{P}/\langle \mathcal{I}, x_1, \dots, x_d \rangle \rightsquigarrow$
  may ignore in residue class $[f] \in \bar{\mathcal{A}}$ all $x^\mu \in \operatorname{supp} f$ with $\operatorname{cls}\mu < d$
- set $\mathcal{H}_{\min} = \{h \in \mathcal{H} \mid \operatorname{cls} h = d + 1\}$ and consider subset $\mathcal{H}_{\text{Soc}}$ of all $h \in \mathcal{H}_{\min}$ such that for no $k > d + 1$ involutive standard representation of $x_k h$ contains non-vanishing *constant* coefficient

Problem: *effective* test for affine Cohen-Macaulay algebra $\mathcal{A} = \mathcal{P}/\mathcal{I}$ to be Gorenstein

Theorem: $\{[h/x_d] \mid h \in \mathcal{H}_{\mathrm{Soc}}\}$ basis of $\operatorname{Soc} \bar{\mathcal{A}}$

Proof: wlog $d = 1$ $\rightsquigarrow$ to prove $\mathcal{I} : \mathfrak{m} = \mathcal{I} + \langle h/x_1 \mid h \in \mathcal{H}_{\mathrm{Soc}} \rangle$

- $f \in \mathcal{I} : \mathfrak{m} \implies x_1 f \in \mathcal{I} \implies$ involutive standard representation $x_1 f = \sum_{h \in \mathcal{H}} P_h h$
  - $\operatorname{cls} h > 1 \implies P_h \in \langle x_1 \rangle$
  - $\operatorname{cls} h = 1 \implies P_h = c_h + x_1 \tilde{P}_h \in \Bbbk[x_1]$

  $\implies f \in \mathcal{I} + \langle h/x_1 \mid h \in \mathcal{H}_{\min} \rangle$
- $\bar{h} \in \mathcal{H}_{\min} \rightsquigarrow$ involutive standard representation $x_k \bar{h} = \sum_{h \in \mathcal{H}} Q_h h$

  Q_h has constant term $\implies h \in \mathcal{H}_{\min}$ and $x_k \bar{h}/x_1 \notin \mathcal{I}$

  $\implies \bar{h}/x_1 \notin \mathcal{I} : \mathfrak{m}$ $h \in \mathcal{H}_{\min}$ and $x_k \bar{h}/x_1 \notin \mathcal{I}$

  $\implies$ only $\bar{h} \in \mathcal{H}_{\mathrm{Soc}}$ contribute to socle

Examples:

- monomial ideal $\mathcal{I} = \langle y^3,\ xy^2,\ x^2 \rangle \lhd \Bbbk[x,y]$
  Pommaret basis $\mathcal{H} = \{y^3,\ \underline{xy^2},\ \underline{x^2y},\ \underline{x^2}\}$

$$y \cdot x^2 = 1 \cdot x^2 y \qquad y \cdot x^2 y = x \cdot xy^2 \qquad y \cdot xy^2 = x \cdot y^3$$

$\operatorname{Soc} \bar{\mathcal{A}} = \langle [xy],\ [y^2] \rangle_{\Bbbk} \implies t = 2,\ \mathcal{A}$ *not* Gorenstein

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- polynomial ideal $\mathcal{I} = \langle z^2 - xy,\ yz,\ y^2,\ xz,\ x^2\rangle \lhd \mathbb{k}[x,y,z]$
  Pommaret basis $\mathcal{H} = \{z^2 - xy,\ yz,\ y^2,\ \underline{xz},\ \underline{x^2y},\ \underline{x^2}\}$

  $$z \cdot xz = x \cdot (z^2 - xy) + 1 \cdot x^2y \qquad y \cdot x^2 = 1 \cdot x^2y$$

  $\operatorname{Soc} \bar{\mathcal{A}} = \langle [xy]\rangle_{\mathbb{k}} \implies t = 1,\ \mathcal{A}$ Gorenstein

  (note: $\mathcal{P}/\operatorname{lt}\mathcal{I}$ *not* Gorenstein, as $[z]$ second socle generator!)