Involutive Bases III

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Overview

- **General Involutive Bases**
- **Basic Algorithms**
- **Pommaret Bases and δ-Regularity**
  - δ- and asymptotic regularity
  - Pommaret vs. Janet division
  - Existence of Pommaret bases
  - Quasi-stable ideals
- **Combinatorial Decompositions and Applications**
- **Syzygy Theory and Applications**

For the rest of the course:

(almost) all appearing polynomials, ideals etc are ***homogeneous***

Convention: $\mathcal{H}$ finite set of polynomials

- $\deg \mathcal{H} = \max \{\deg h \mid h \in \mathcal{H}\}$
- $\operatorname{cls} \mathcal{H} = \min \{\operatorname{cls} h \mid h \in \mathcal{H}\}$

Problem: not every ideal $\mathcal{I} \lhd \Bbbk[X]$ has *finite* Pommaret basis (existence guaranteed only for *zero-dimensional* ideals $\rightsquigarrow$ later)

Claim: only a problem of the chosen *variables* $X = \{x_1, \dots, x_n\}$

Def: variables X δ-*regular* for $\mathcal{I}$ and term order $\prec$ $\rightsquigarrow$ $\mathcal{I}$ possesses *finite* Pommaret basis for $\prec$

δ-Regularity

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Def: variables X δ*-regular* for $\mathcal{I}$ and term order $\prec$ $\leadsto$ $\mathcal{I}$ possesses *finite* Pommaret basis for $\prec$

Idea: assume term order $\prec$ defined on *exponent vectors* $\leadsto$ *linear transformation* $Y = AX$ with non-singular matrix $A \in \Bbbk^{n\times n}$ transforms polynomial $f \in \Bbbk[X]$ into new polynomial $\tilde{f} \in \Bbbk[Y]$ $\leadsto$ sort terms with same order as before — *and hope that things get better. . .*

Observation: δ-regularity is an *asymptotic* property

Lemma: ideal $\mathcal{I} \lhd \Bbbk[X]$

(i) $\mathcal{H}$ Pommaret basis of $\mathcal{I}$, $q \geq \deg \mathcal{H}$ $\Longrightarrow$

$$\mathcal{H}_q = \{x^\mu h \mid h \in \mathcal{H},\ \deg(x^\mu h) = q,\ x^\mu \in \mathbb{T}(x_1, \dots, x_{\operatorname{cls} h})\}$$

Pommaret basis of *truncated* ideal $\mathcal{I}_{\geq q}$

(ii) truncation $\mathcal{I}_{\geq q}$ has finite Pommaret basis for some $q \in \mathbb{N}_0^n$ $\Longrightarrow$ $\mathcal{I}$ has finite Pommaret basis

Proof:

(i) straightforward computation

(ii) take finite Pommaret basis $\hat{\mathcal{H}}$ of truncated ideal $\mathcal{I}_{\geq q}$
add $\Bbbk$-*linear* bases of $\mathcal{I}_r$ for all lower degrees $0 \leq r < q$ $\rightsquigarrow$
weak Pommaret basis of $\mathcal{I}$

Consider finite, Pommaret head autoreduced set $\mathcal{F} \subset \Bbbk[X]$ $\rightsquigarrow$
linear transformation $Y = AX$ yields new set $\tilde{\mathcal{F}} \subset \Bbbk[Y]$ $\rightsquigarrow$
Pommaret head autoreduction $\rightsquigarrow$ final set $\tilde{\mathcal{F}}^{\Delta} \subset \Bbbk[Y]$

introduce *"Hilbert functions"* for $\mathcal{I} = \langle \mathcal{F} \rangle$ and involutive spans:

- $h_{\mathcal{I}}(r) = \dim_{\Bbbk} \mathcal{I}_r = \dim_{\Bbbk} \tilde{\mathcal{I}}_r$
- $h_{\mathcal{F},P,\prec}(r) = \dim_{\Bbbk} \big(\langle \mathcal{F} \rangle_{P,\prec}\big)_r \leq h_{\mathcal{I}}(r)$
- $h_{\tilde{\mathcal{F}}^{\Delta},P,\prec}(r) = \dim_{\Bbbk} \big(\langle \tilde{\mathcal{F}}^{\Delta} \rangle_{P,\prec}\big)_r \leq h_{\mathcal{I}}(r)$

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Def: $\mathcal{F} \subset \Bbbk[X]$ finite, Pommaret head autoreduced set;
variables X *asymptotically regular* for $\mathcal{F}$ and term order $\prec$ $\rightsquigarrow$

$$\forall A \in \Bbbk^{n \times n} \text{ non-singular, } r \gg 0 \,:\, h_{\mathcal{F},P,\prec}(r) \geq h_{\tilde{\mathcal{F}}^{\Delta},P,\prec}(r)$$

Ex: $\mathcal{F} = \{f = x_1x_2\} \subset \Bbbk[x_1, x_2]$, $\prec = \prec_{\text{degrevlex}}$

- $\operatorname{cls} f = 1 \Longrightarrow X_{P,\prec}(f) = \{x_1\} \Longrightarrow$

$$\forall r \geq 2 : h_{\mathcal{F},P,\prec}(r) = 1 < h_{\mathcal{I}}(r) = r - 1$$

- transformation: $x_1 = y_1 + y_2$, $x_2 = y_2 \rightsquigarrow$

$$\tilde{\mathcal{F}}^\Delta = \{\tilde{f} = y_2^2 + y_1y_2\} \subset \Bbbk[y_1, y_2]$$

$\operatorname{cls} y_2^2 = 2 \Longrightarrow Y_{P,\prec}(\tilde{f}) = \{y_1, y_2\} = Y \Longrightarrow$

$$\forall r > 2 : h_{\tilde{\mathcal{F}}^\Delta,P,\prec}(r) = h_{\mathcal{I}}(r) > h_{\mathcal{F},P,\prec}(r)$$

X *not* asymptotically regular for $\mathcal{F}$ and $\prec$ *but*
Y asymptotically regular for $\tilde{\mathcal{F}}$ and $\prec$ (and δ-regular for $\langle\tilde{\mathcal{F}}\rangle$)

Note: asymptotic and δ-regularity generally independent properties

- δ-*regularity* concerned with $\operatorname{lt}\mathcal{I}$
- *asymptotic regularity* concerned with $\langle \operatorname{lt}\mathcal{F}\rangle \subseteq \operatorname{lt}\mathcal{I}$

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Lemma: X δ-regular for $\mathcal{I}$ and $\prec$, $\mathcal{H}$ Pommaret basis of $\mathcal{I}$ for $\prec$ $\Longrightarrow$ X asymptotically regular for $\mathcal{H}$ and $\prec$

Proof: $h_{\mathcal{H},P,\prec} = h_{\mathcal{I}}$

Asymptotic Regularity

Choosing *“reference variables”* X, we can identify each set of variables $Y = AX$ with the matrix $A \in \mathbb{k}^{n\times n}$

Prop: $\mathcal{F}$ involutively head autoreduced, $\prec$ fixed term order $\Longrightarrow$ variables asymptotically regular for $\mathcal{F}$ and $\prec$ form *Zariski open* set in $\mathbb{k}^{n\times n}$

Proof: consider transformation with *undetermined* matrix A $\rightsquigarrow$ leading coefficients in $\tilde{\mathcal{F}}^{\Delta}$ *polynomials* in entries of A $\rightsquigarrow$ asymptotically *singular* variables characterised by *vanishing* of certain leading coefficients $\rightsquigarrow$ correspond to variety in $\mathbb{k}^{n\times n}$

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Theoretical solution of problem of asymptotic regularity $\rightsquigarrow$ perform linear *random* transformation of variables (though *no* guarantee of asymptotic regularity of new variables)

Practically useless, as all *sparsity* in $\mathcal{F}$ destroyed by transformation $\rightsquigarrow$ all subsequent computations much more expensive

Pommaret vs. Janet Division

Pommaret and Janet division are defined very differently

but: yield often same multiplicative variables for finite sets $\mathcal{T} \subset \mathbb{T}(X)$

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Cor: $\mathcal{T}_P = \mathcal{T} \setminus \{t \in \mathcal{T} \mid \exists t \neq s \in \mathcal{T} : s \,|_P\, t\} \Longrightarrow$

$$\langle \mathcal{T} \rangle_J \subseteq \langle \mathcal{T}_P \rangle_J$$

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Cor: $\mathcal{H}$ *Pommaret* basis of $\mathcal{I}$ and $\prec$ $\Longrightarrow$ $\mathcal{H}$ also *Janet* basis

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Proof: $X_{P,\prec}(f) \subset X_{J,\mathcal{F},\prec}(f) \;\wedge\; \operatorname{cls} f = k \implies$
$\exists\, x_\ell \in X_{J,\mathcal{F},\prec}(f) : \ell > k \;\rightsquigarrow$
transformation $x_k = y_k + c y_l$ and $x_i = y_i$ else $\implies$
$\operatorname{lt} f$ transforms into polynomial where leading term has class $> k \;\rightsquigarrow$
choose c such that leading coefficient does not vanish $\implies$
$h_{\tilde{\mathcal{F}}^\Delta,P,\prec}$ asymptotically larger than $h_{\mathcal{F},P,\prec}$ for almost all $c \in \Bbbk$

Existence of Pommaret Bases

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Thus: *necessary* (but *not* sufficient) criterion for asymptotic regularity

$$\forall\, f \in \mathcal{F} : X_{P,\prec}(f) = X_{J,\mathcal{F},\prec}(f)$$

Assume $|\mathbb{k}| = \infty$ (or $\mathbb{k}$ “sufficiently” large)

Theorem: *every* ideal $\mathcal{I} \subseteq \mathbb{k}[X]$ possesses finite Pommaret basis in suitable variables X

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Proof:

- use first corollary above to prove existence of *Pommaret* autoreduced *Janet* basis (apply completion algorithm with Janet division and Pommaret autoreductions)
- apply *modified* algorithm with *undetermined* variables $\rightsquigarrow$ only *finitely* many intermediate bases $\mathcal{H}_i$
- choose variables Y asymptotically regular for *all* $\mathcal{H}_i$ (always possible by *genericity* of asymptotic regularity)
- result *simultaneously* Janet and Pommaret basis of transformed ideal $\tilde{\mathcal{I}} \subseteq \mathbb{k}[Y]$ by second corollary above

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Cor: variables δ-regular for $\mathcal{I}$ form Zariski open set in $\mathbb{k}^{n\times n}$

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Two possible algorithmic realisations:

- check criterion after each completion step $\rightsquigarrow$
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  if not simultaneously Pommaret basis perform transformation
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For the $\mathcal{T}$-$\mathcal{Q}$ algorithm the *second* strategy is almost always better: even in δ-regular variables it requires for the Pommaret division generally much more non-multiplicative products.

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General problem: prove that *finite* number of transformations suffices

(solvable for $\prec = \prec_{\text{degrevlex}}$ with more theory)

Ex: $\mathcal{F} = \{z^2 - y^2 - 2x^2,\ xz + yx,\ yz + y^2 + x^2\}\,, \quad \prec = \prec_{\text{degrevlex}}$

- same multiplicative variables for Janet and Pommaret division
  but: variables not asymptotically regular for $\mathcal{F}$
  (consider transformation $\tilde{x} = z$, $\tilde{y} = y + z$, $\tilde{z} = x$)
- first completion step: analyse $y(xz + xy) \quad \rightsquigarrow \quad -x^3$
  new basis $\hat{\mathcal{F}} = \mathcal{F} \cup \{x^3\} \quad \rightsquigarrow \quad$ variables *not* asymptotically regular

$$X_{P,\prec}(x^3) = \{x\} \subsetneq X_{J,\hat{\mathcal{F}},\prec}(x^3) = \{x, y\}$$

  (completion does not terminate: $x^3y, x^3y^2, \dots$)
- $\hat{\mathcal{F}}$ Janet but *not* Pommaret basis: $\deg \hat{\mathcal{F}} = 3$, $|\hat{\mathcal{F}}| = 4$
- perform above coordinate transformation

$$\tilde{\mathcal{F}}^{\Delta} = \{\tilde{z}^2 - \tilde{x}\tilde{y},\ \tilde{y}\tilde{z} - \tilde{x},\ \tilde{y}^2 - \tilde{z}\}$$

  Janet *and* Pommaret basis with $\deg \tilde{\mathcal{F}}^{\Delta} = 2$, $|\tilde{\mathcal{F}}^{\Delta}| = 3$

For *monomial* ideals variable transformations uninteresting, as transformed ideal generally no longer monomial $\rightsquigarrow$
Pommaret division distinguishes class of monomial ideals

Def: monomial ideal $\mathcal{I}$ *quasi-stable* $\rightsquigarrow$ $\mathcal{I}$ has finite Pommaret basis

Remark: many alternative names for these ideals in the literature

- *ideals of nested type* (see below)
- *ideals of Borel type* (Borel fixed ideals are quasi-stable)
- *weakly stable ideals* (stable ideals are quasi-stable)
- *ideals in strong Noether position*
  (quasi-stable ideals are in Noether position, but not every ideal in Noether position is quasi-stable — see next lecture)

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Prop:

- $\mathcal{I}_1, \mathcal{I}_2$ quasi-stable $\Longrightarrow$ $\mathcal{I}_1 + \mathcal{I}_2, \mathcal{I}_1 \cdot \mathcal{I}_2, \mathcal{I}_1 \cap \mathcal{I}_2$ quasi-stable
- $\mathcal{I}$ quasi-stable, $\mathcal{J}$ arbitrary $\Longrightarrow$ $\mathcal{I} : \mathcal{J}$ quasi-stable

Proof: Pommaret bases $\mathcal{H}_k$ of $\mathcal{I}_k$

- $\mathcal{H}_1 \cup \mathcal{H}_2$ *weak* Pommaret basis of $\mathcal{I}_1 + \mathcal{I}_2$
- $\{h_1 h_2 \mid h_k \in \mathcal{H}_k\}$ *weak* Pommaret basis of $\mathcal{I}_1 \cdot \mathcal{I}_2$
- $\{\operatorname{lcm}(h_1, h_2) \mid h_k \in \mathcal{H}_k\}$ *weak* Pommaret basis of $\mathcal{I}_1 \cap \mathcal{I}_2$

(theoretical application of *continuity* of Pommaret division)

Quasi-stability is an *intrinsic* algebraic property!

Prop: equivalent are

(i) $\mathcal{I}$ quasi-stable

(ii) $\mathcal{I} : x_1^\infty \subseteq \mathcal{I} : x_2^\infty \subseteq \cdots \subseteq \mathcal{I} : x_n^\infty$
(with $\mathcal{I} : x_k^\infty = \mathcal{P}$ for all $k \geq \dim \mathcal{P}/\mathcal{I}$)

(iii) $\forall k : \mathcal{I} : x_k^\infty = \mathcal{I} : \langle x_k, \dots, x_n \rangle^\infty$

(iv) every *associated prime* of $\mathcal{P}/\mathcal{I}$ is of the form $\langle x_k, \dots, x_n \rangle$

(v) x_1 is *non zero divisor* in $\mathcal{P}/\mathcal{I}^{\text{sat}}$ and x_{k+1} is *non zero divisor* in $\mathcal{P}/\langle \mathcal{I}, x_1, \dots, x_k \rangle^{\text{sat}}$ for every $k > 0$

(vi) $x^\mu \in \mathcal{I}$ and $\mu_i > 0$ for some $1 \leq i < n \implies$
$\forall\, 0 < r \leq \mu_i,\ i < j \leq n\ \exists\, s \geq 0 : x_j^s x^\mu / x_i^r \in \mathcal{I}$

(recall: *saturation* $\mathcal{I}^{\text{sat}} = \mathcal{I} : \langle x_1, \dots, x_n \rangle^\infty$)

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- **(ii)** and **(iii)** easily *effectively* verifiable
- both allow check whether *permutation* suffices to obtain δ-regular variables
- simple *first* step: search for generators of the form x_k^e $\rightsquigarrow$ renumber corresponding variables as $x_n, x_{n-1}, \dots$

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Cor: $\mathcal{I}$ *not* quasi-stable, $\mathcal{B}$ finite, Pommaret autoreduced monomial basis
$\implies \exists\, t \in \mathcal{B} : X_P(t) \subsetneq X_{J,\mathcal{B}}(t)$

(thus variables always asymptotically singular for not quasi-stable ideal)

Simple examples of quasi-stable ideals:

(i) *irreducible ideals*

$$\mathcal{I} = \langle x_{i_r}^{\ell_{i_r}}, \dots, x_{i_2}^{\ell_{i_2}}, x_{i_1}^{\ell_{i_1}} \rangle \quad \text{where } i_1 < \cdots < i_r$$

recall from Lecture 1:
$\mathcal{I}$ quasi-stable $\iff$ $i_r = n,\ i_{r-1} = n-1, \dots,\ i_1 = n-r+1$

Pommaret basis $\mathcal{H}$ of $\mathcal{I}$ then consists of all terms $x_{i_j}^{\ell_{i_j}} x_{i_j+1}^{k_{i_j+1}} \cdots x_n^{k_n}$ with $\forall m > i_j : k_m < \ell_m$

Simple examples of quasi-stable ideals:

(ii) *zero-dimensional ideals*

consider arbitrary zero-dimensional monomial ideal $\mathcal{J}$

- $\mathcal{J}$ contains irreducible ideal $\mathcal{I} = \langle {x_1}^{\ell_1}, \dots, x_n^{\ell_n} \rangle \subseteq \mathcal{J}$
- take (weak) Pommaret basis $\mathcal{H}_\mathcal{I}$ of $\mathcal{I}$
- add all monomials $x^\mu \in \mathcal{J} \setminus \mathcal{I}$ (finitely many!)
- obtain *weak* Pommaret basis $\mathcal{H}_\mathcal{J}$ of $\mathcal{J}$ $\Longrightarrow$ $\mathcal{J}$ quasi-stable

Simple examples of quasi-stable ideals:

(iii) *(reverse) lexicographic* ideals

monomial ideal $\mathcal{I}$ (reverse) lexicographic $\rightsquigarrow$
$\forall\, q \geq 0\, \exists\, r_q \geq 0$: component $\mathcal{I}_q$ generated by r_q greatest terms of order q

Def: monomial ideal $\mathcal{I} \lhd \mathbb{k}[x_1, \dots, x_n]$ *stable* $\rightsquigarrow$

$$\forall \text{ terms } t \in \mathcal{I} \; \forall\, n \geq \ell > k = \operatorname{cls} t \; : \; \frac{x_\ell}{x_k} t \in \mathcal{I}$$

(definition independent of $\operatorname{char} \mathbb{k}$ — opposed to Borel fixed!)

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Prop: $\mathcal{I}$ stable $\Longleftrightarrow$ *minimal* basis $\mathcal{B}$ of $\mathcal{I}$ Pommaret basis

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Prop: $\mathcal{I}$ stable $\iff$ *minimal* basis $\mathcal{B}$ of $\mathcal{I}$ Pommaret basis

Prop: $\mathcal{I}$ quasi-stable $\Longrightarrow$ $\mathcal{I}_{\geq q}$ stable for $q \gg 0$

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Remark: in δ-regular coordinates leading ideal $\operatorname{lt} \mathcal{I}$ of any polynomial ideal $\mathcal{I} \subseteq \mathcal{P}$ always quasi-stable; in general $\operatorname{lt} \mathcal{I}$ *not* Borel fixed and thus *not* the *generic initial ideal* $\operatorname{gin} \mathcal{I}$; but $\operatorname{lt} \mathcal{I}$ shares almost all properties of $\operatorname{gin} \mathcal{I}$

$\mathcal{V}$ arbitrary n-dimensional $\mathbb{k}$-linear space
$\mathcal{M}$ finitely generated graded $S\mathcal{V}$-module

Lemma: for any degree $q > 0$ and any homogeneous element $m \in \mathcal{M}$ the following statements are equivalent

(i) $\operatorname{Ann}(m) = S_+\mathcal{V} \implies m \in \mathcal{M}_{<q}$

(ii) $(\exists v \in \mathcal{V} : v \cdot m = 0) \implies m \in \mathcal{M}_{<q}$

(iii) for all $v \in \mathcal{V}$ outside a finite number of proper linear subspaces
$v \cdot m = 0 \implies m \in \mathcal{M}_{<q}$

Proof: **(iii)** $\Longrightarrow$ **(ii)** $\Longrightarrow$ **(i)**: obvious

(i) $\Longrightarrow$ **(iii)**: let $\mathcal{A} = \{m \in \mathcal{M} \mid \operatorname{Ann}(m) = S_+\mathcal{V}\}$ and choose $\mathcal{K}$ with
$\mathcal{M}_{<q} = \mathcal{A} \oplus \mathcal{K} \quad \rightsquigarrow \quad \overline{\mathcal{M}} = \mathcal{K} \oplus \bigoplus_{r \geq q} \mathcal{M}_r$
$\operatorname{Ass}\overline{\mathcal{M}}$ finite set and $S_+\mathcal{V} \notin \operatorname{Ass}\overline{\mathcal{M}} \implies$
$\forall\, \mathfrak{p} \in \operatorname{Ass}\overline{\mathcal{M}} : \mathfrak{p} \cap \mathcal{V}$ proper subspace of $\mathcal{V}$
$v \in \mathcal{V} \setminus \bigcup_{\mathfrak{p} \in \operatorname{Ass}\overline{\mathcal{M}}} \mathfrak{p} \implies \forall\, m \in \overline{\mathcal{M}} : v \cdot m \neq 0$

$\mathcal{V}$ arbitrary n-dimensional $\mathbb{k}$-linear space
$\mathcal{M}$ finitely generated graded $S\mathcal{V}$-module

Def: $v \in \mathcal{V}$ *quasi-regular* at degree q $\rightsquigarrow$ $v \cdot m = 0 \implies m \in \mathcal{M}_{<q}$
sequence $(v_1, \dots, v_k) \in \mathcal{V}^k$ *quasi-regular* at degree q $\rightsquigarrow$
v_i quasi-regular at degree q for $\mathcal{M}/\langle v_1, \dots, v_{i-1}\rangle\mathcal{M}$

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Consider ideal $\mathcal{I} \lhd S\mathcal{V}$ and corresponding factor ring $\mathcal{A} = S\mathcal{V}/\mathcal{I}$.

Theorem: $\mathbb{k}$-linear basis $(x_1, \dots, x_n)$ of $\mathcal{V}$ δ-regular for ideal $\mathcal{I} \lhd \mathbb{k}[x_1, \dots, x_n] \cong S\mathcal{V}$ in the sense that Pommaret basis $\mathcal{H}$ exists for `degrevlex` with $\deg \mathcal{H} = q$ $\iff$
$(x_1, \dots, x_n)$ quasi-regular for $\mathcal{A}$ at degree q but not at any lower degree