Involutive Bases I

Werner M. Seiler
Institut für Mathematik
Universität Kassel

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- **General Involutive Bases**
  - Shortcomings of Gröbner Bases
  - Involutive Divisions with Examples
  - Involutive Bases: The Monomial Case
  - Involutive Bases: The Polynomial Case
  - Elementary Properties
- **Basic Algorithms**
- **Pommaret Bases and δ-Regularity**
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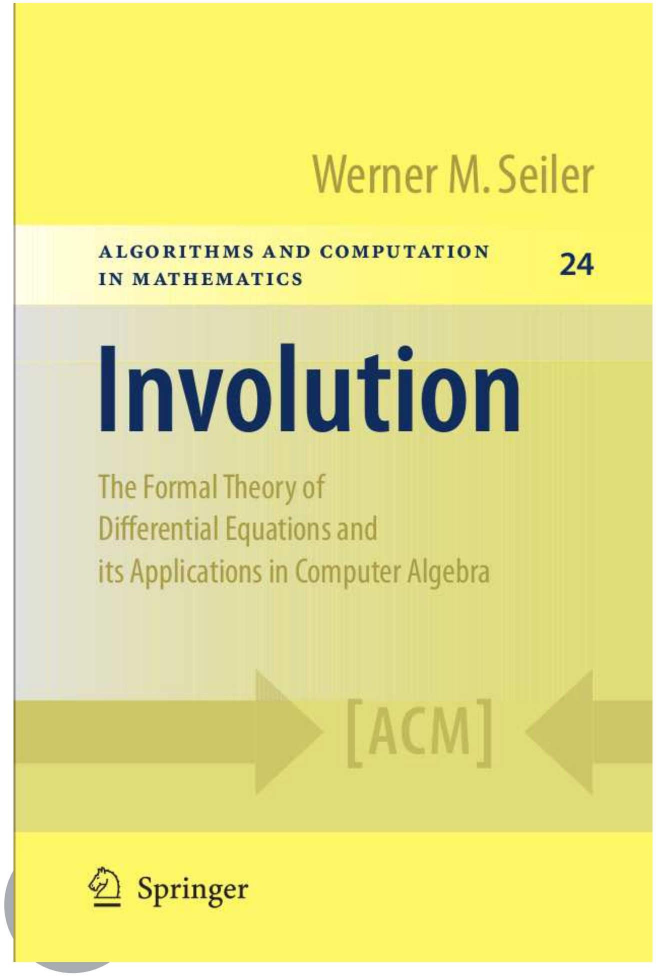

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Shortcomings of Gröbner bases:

- definition not **intrinsic**:
  - arbitrary choice of variables and term order
  - in principle: $\mathcal{P} \cong S\mathcal{V}$ for n-dimensional $\mathbb{k}$-linear space $\mathcal{V}$ with basis $\{x_1, \dots, x_n\}$
  - change of basis or term order can affect Gröbner basis drastically
- purely **technical** definition $\rightsquigarrow$ *algebraic* properties of ideal hardly enter
- determination of **algebraic structure** of ideal requires additional computations (often *several* Gröbner bases)

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Example: consider

$$\mathcal{I} = \langle z^8 - wxy^6,\ y^7 - x^6 z,\ yz^7 - wx^7 \rangle \lhd \mathbb{k}[w, x, y, z]$$

(reduced) Gröbner basis for `degrevlex`

Consider $\mathcal{I}$ as ideal in $\mathbb{k}[w, y, x, z]$:
(reduced) Gröbner basis for `degrevlex`

$$\{y^7 - x^6 z,\ yz^7 - wx^7,\ z^8 - wxy^6,\ y^8 z^6 - wx^{13},\\ y^{15} z^5 - wx^{19},\ y^{22} z^4 - wx^{25},\ y^{29} z^3 - wx^{31},\\ y^{36} z^2 - wx^{37},\ y^{43} z - wx^{43},\ y^{50} - wx^{49}\}$$

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Involutive Bases:

- Special type of (generally *non-reduced*) Gröbner bases with additional **combinatorial** properties
- Provide alternative **algorithm** for construction of Gröbner bases (often *superior* to Buchberger algorithm)
- **Pommaret bases** for `degrevlex` particularly interesting for applications in *algebraic geometry* $\rightsquigarrow$ many characteristics (e. g. degree of basis) *intrinsically* determined by **homological** invariants of ideal
- **Pommaret bases** not only **computationally** of interest; allow for **theoretical** applications, too.

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Differential Equations

Jacobi $\leq$1862

Riquier 1890–1910

(Cartan 1900–1930)

Janet 1920

Commutative Algebra

(Gröbner *1899)

(Gordan 1900)

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Differential Equations

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Riquier 1890–1910
(Cartan 1900–1930)
Janet 1920

Spencer 1962
Quillen 1964
Goldschmidt 1965
Pommaret 1978

Commutative Algebra

*(Gröbner *1899)*
(Gordan 1900)

Rees 1956 (P)
Hironaka 1964/77 (P)
Buchberger 1965
Grauert 1972 (P)
Stanley 1978
Baclawski/Garcia 1981

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Differential Equations	Commutative Algebra
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Riquier 1890–1910	*(Gröbner *1899)*
(Cartan 1900–1930)	(Gordan 1900)
Janet 1920	
	Rees 1956 (P)
Spencer 1962	Hironaka 1964/77 (P)
Quillen 1964	Buchberger 1965
Goldschmidt 1965	Grauert 1972 (P)
Pommaret 1978	Stanley 1978
	Baclawski/Garcia 1981

Amasaki 1990 (P)
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- field $\Bbbk$ of arbitrary characteristic
  (later: "sufficiently large" for Pommaret bases)
- set of variables $X = \{x_1, \dots, x_n\}$
  (ordering of variables relevant for many purposes!)
- polynomial ring $\mathcal{P} = \Bbbk[X]$ $\rightsquigarrow$
  ideals $\mathcal{I} \lhd \mathcal{P}$, submodules $\mathcal{M} \subseteq \mathcal{P}^m$ of free $\mathcal{P}$-module
- subset of variables $X' \subseteq X$ $\rightsquigarrow$
  $\mathbb{T}(X')$ monoid of terms containing only variables in X'
- multi indices $\mu, \nu \in \mathbb{N}_0^n$ $\rightsquigarrow$ terms $x^\mu, x^\nu \in \mathbb{T}(X)$

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Caution: the following conventions are *non-standard*; the standard ones are obtained by reverting the ordering of the variables

$$x_1, x_2, \dots, x_n \quad \longleftrightarrow \quad x_n, \dots, x_2, x_1$$

- $\mu = [\mu_1, \dots, \mu_n] \quad \rightsquigarrow \quad$ *class* $\operatorname{cls}\mu = \min\{k \mid \mu_k \neq 0\}$
- *lexicographic term order*
  $\mu \prec_{\text{lex}} \nu \quad \Longleftrightarrow \quad$ *last* non-vanishing entry of $\mu - \nu$ *negative*
- *degree reverse lexicographic term order* (for $|\mu| = |\nu|$)
  $\mu \prec_{\text{degrevlex}} \nu \quad \Longleftrightarrow \quad$ *first* non-vanishing entry of $\mu - \nu$ *positive*
- `degrevlex` only *class respecting* term order

Lemma: Assume $\prec$ degree compatible term order and for all *homogeneous* polynomials f and all $1 \leq k \leq n$

$$\operatorname{lt} f \in \langle x_1, \dots, x_k\rangle \quad \Longleftrightarrow \quad f \in \langle x_1, \dots, x_k\rangle\,.$$

Then $\prec = \prec_{\text{degrevlex}}$.

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Basic idea: every generator in involutive basis may only be multiplied by polynomials in its *multiplicative variables* $\rightsquigarrow$ *two* **ingredients** required:

- *term order* (like any Gröbner basis)
- *involutive division* for assignment of multiplicative variables (based on the leading terms of the generators)

Difficulty: assignment is generally made "context sensitive": multiplicative variables for term $t \in \mathbb{T}(X)$ not absolutely defined, but t must always be considered in "context" of finite set $\mathcal{T} \subset \mathbb{T}(X)$ (say, all leading terms in a basis) containing t

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Def: **involutive division** L on $\mathbb{T}(X)$ $\rightsquigarrow$
rule to assign to every term t in any finite set $\mathcal{T} \subset \mathbb{T}(X)$ **multiplicative variables** $X_{L,\mathcal{T}}(t)$ and thus **involutive cone** $\mathcal{C}_{L,\mathcal{T}}(t) = \mathbb{T}\big(X_{L,\mathcal{T}}(t)\big) \cdot t$ such that:

(i) $s, t \in \mathcal{T}$ with $\mathcal{C}_{L,\mathcal{T}}(s) \cap \mathcal{C}_{L,\mathcal{T}}(t) \neq \emptyset \implies$
$\mathcal{C}_{L,\mathcal{T}}(s) \subseteq \mathcal{C}_{L,\mathcal{T}}(t)$ or vice versa

(ii) $\mathcal{S} \subseteq \mathcal{T} \implies \forall s \in \mathcal{S} : X_{L,\mathcal{T}}(s) \subseteq X_{L,\mathcal{S}}(s)$

Def: $s \in \mathbb{T}(X)$ **involutively divisible** by $t \in \mathcal{T}$ (written $t \,|_{L,\mathcal{T}}\, s$) $\rightsquigarrow$
$s \in \mathcal{C}_{L,\mathcal{T}}(t)$

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Example: *Thomas division* T $\rightsquigarrow$

$$x_i \in X_{T,\mathcal{T}}(t) \iff \deg_{x_i} t = \max\{\deg_{x_i} s \mid s \in \mathcal{T}\}$$

(independent of ordering of variables;
not relevant for practice; sometimes theoretically useful)

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Example: *Janet division* J ⇝

given $\mathcal{T} \subset \mathbb{T}(X)$, introduce for arbitrary $d_i \in \mathbb{N}_0^n$ subsets

$$(d_k, \dots, d_n) = \{x^\mu \in \mathcal{T} \mid \mu_k = d_k, \dots, \mu_n = d_n\} \subseteq \mathcal{T}$$

then for term $t = x^\mu \in \mathcal{T}$

- $x_n \in X_{J,\mathcal{T}}(t) \iff \deg_{x_n} t = \max\{\deg_{x_n} s \mid s \in \mathcal{T} = ()\}$
- for $1 \leq k < n$: $x_k \in X_{J,\mathcal{T}}(t) \iff$
  $\deg_{x_k} t = \max\{\deg_{x_k} s \mid s \in (\mu_{k+1}, \dots, \mu_n)\}$

(“refinement” of Thomas division; depends on ordering of variables)

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Example: *Pommaret division* P $\rightsquigarrow$

$$t = x^{\mu} \in \mathcal{T}, \operatorname{cls}\mu = k \quad \Longrightarrow \quad X_P(t) = \{x_1, \dots, x_k\}$$

(depends on ordering of variables, too)

global division: no dependence on $\mathcal{T}$

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$t = x^\mu \in \mathcal{T}, \operatorname{cls}\mu = k \implies X_P(t) = \{x_1, \dots, x_k\}$

(depends on ordering of variables, too)

global division: no dependence on $\mathcal{T}$

Example: bizarre global division on $\mathbb{T}(x, y, z)$

$$X_L(1) = \{x, y, z\}$$
$$X_L(x) = \{x, z\}, \quad X_L(y) = \{x, y\}, \quad X_L(z) = \{y, z\},$$
$$X_L(t) = \emptyset \text{ for all other } t \in \mathbb{T}(x, y, z)$$

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Example: $\mathcal{T} = \{z^3,\, y^2z,\, yz^2,\, xz^2,\, xyz\} \subset \mathbb{T}(x, y, z)$

multiplicative variables for different involutive divisions

	z^3	y^2z	yz^2	xz^2	xyz
T	z	y	–	x	x
J	x, y, z	x, y	x, y	x	x
P	x, y, z	x, y	x, y	x	x

(note: despite very different definitions, Janet and Pommaret division yield here same multiplicative variables $\leadsto$ more in Lecture 3!)

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Def: $\mathcal{T} \subset \mathbb{T}(X)$ finite, involutive division L

- *involutive span of* $\mathcal{T}$ $\rightsquigarrow$ $\langle\mathcal{T}\rangle_L = \langle \bigcup_{t\in\mathcal{T}} \mathcal{C}_{L,\mathcal{T}}(t)\rangle_{\Bbbk}$
- $\mathcal{T}$ *weakly involutive* $\rightsquigarrow$ $\langle\mathcal{T}\rangle_L = \langle\mathcal{T}\rangle$
- $\mathcal{T}$ *(strongly) involutive* $\rightsquigarrow$ additionally

$$\forall t \neq t' \in \mathcal{T} \; : \; \mathcal{C}_{L,\mathcal{T}}(t) \cap \mathcal{C}_{L,\mathcal{T}}(t') = \emptyset$$

(provides $\Bbbk$-linear *direct sum decomposition* of ideal $\langle\mathcal{T}\rangle$!)

- $\mathcal{T} \subseteq \hat{\mathcal{T}}$ (finite) *weakly involutive completion* $\rightsquigarrow$ $\langle\hat{\mathcal{T}}\rangle_L = \langle\mathcal{T}\rangle$
- $\mathcal{T}$ *(weakly) involutive basis* of monomial ideal $\mathcal{I} \lhd \mathcal{P}$ $\rightsquigarrow$ $\mathcal{T}$ (weakly) involutive and $\langle\mathcal{T}\rangle_L = \mathcal{I}$

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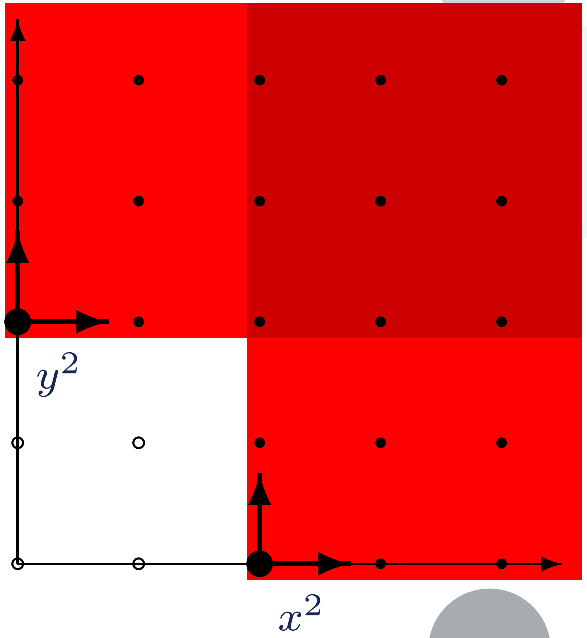

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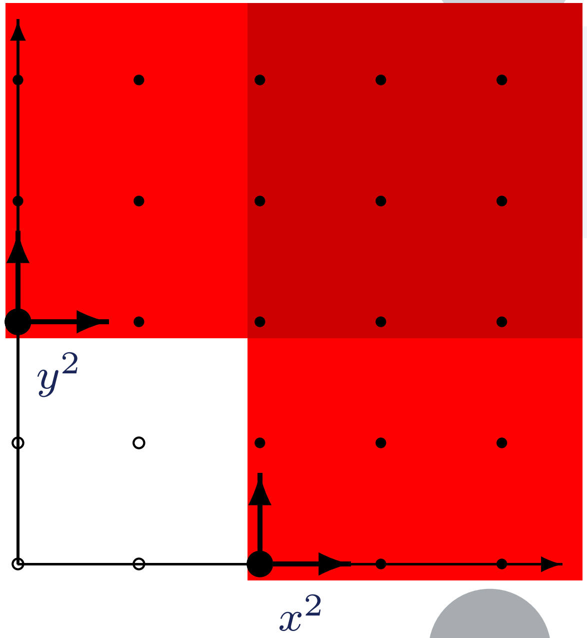

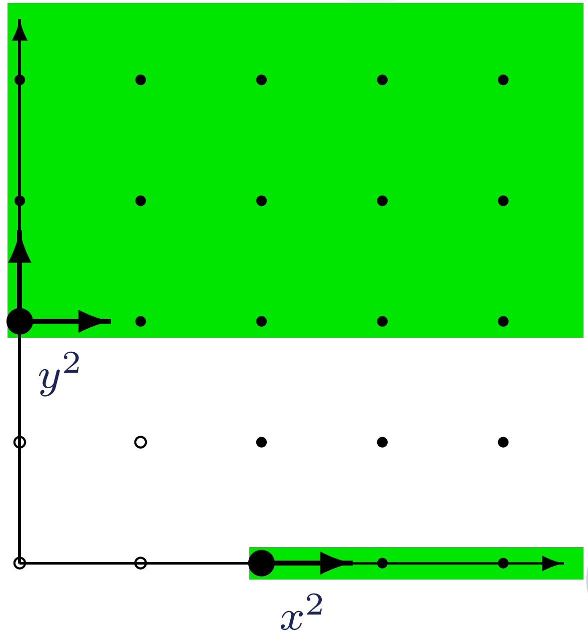

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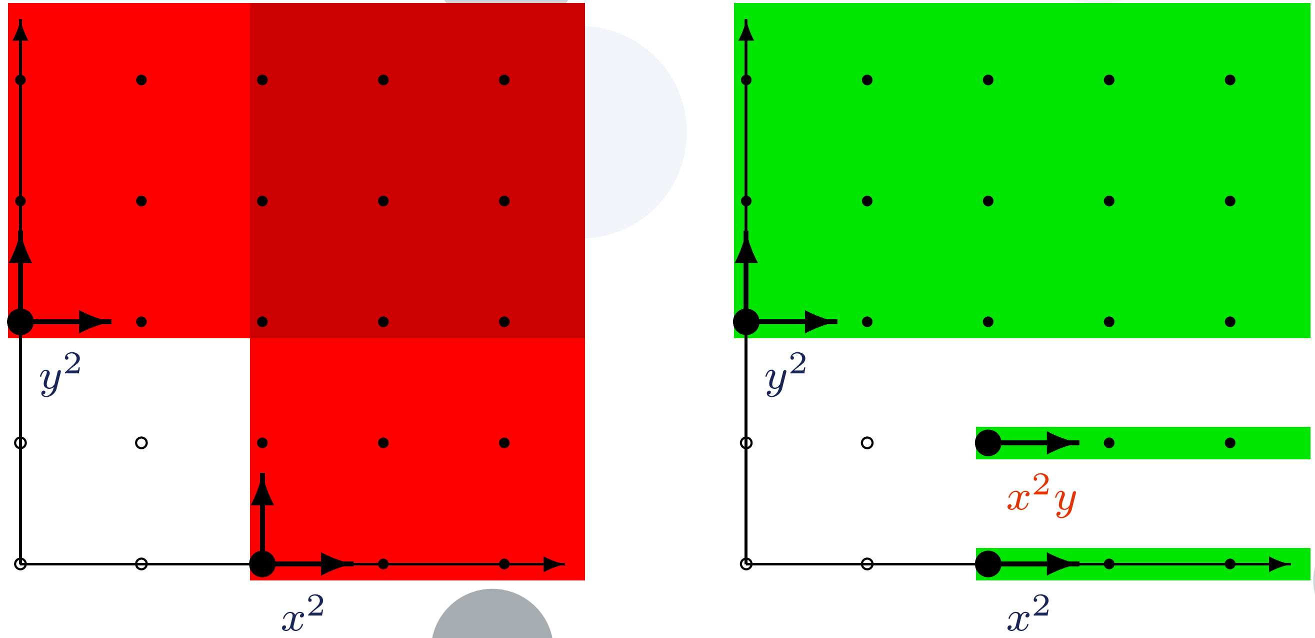

$\Bbbk$-linear isomorphism: *(Stanley decomposition ⇝ Lecture 4)*

$$\mathcal{I} = \langle x^2, y^2 \rangle \cong \Bbbk[x, y] \cdot y^2 \oplus \Bbbk[x] \cdot x^2 \oplus \Bbbk[x] \cdot x^2 y$$

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Prop: $\mathcal{T}$ *weakly* involutive basis $\Longrightarrow$ $\exists\, \mathcal{T}' \subseteq \mathcal{T}$ *strongly* involutive basis

Proof:

$\mathcal{T}$ weakly but not strongly involutive basis $\Longrightarrow$

$\exists\, s \neq t \in \mathcal{T} \,:\, \mathcal{C}_{L,\mathcal{T}}(s) \cap \mathcal{C}_{L,\mathcal{T}}(t) \neq \emptyset \quad \overset{\text{(i)}}{\Longrightarrow}$

(wlog) $\mathcal{C}_{L,\mathcal{T}}(s) \subseteq \mathcal{C}_{L,\mathcal{T}}(t) \quad \rightsquigarrow$ set $\mathcal{T}' = \mathcal{T} \setminus \{s\}$

$\overset{\text{(ii)}}{\Longrightarrow}$ $\mathcal{T}'$ still weakly involutive basis $\rightsquigarrow$ iterate

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Example: minimal basis of *irreducible* monomial ideal of the form

$$\mathcal{T} = \{x_{i_r}^{\ell_{i_r}}, \dots, x_{i_2}^{\ell_{i_2}}, x_{i_1}^{\ell_{i_1}}\}$$

with $1 \leq r \leq n$ generators sorted according to $i_r > \cdots > i_2 > i_1$

$\mathcal{T}$ has finite Pommaret completion $\hat{\mathcal{T}}$ $\iff$
$i_r = n,\ i_{r-1} = n-1, \dots,\ i_1 = n-r+1$ (i. e. no "gaps")
(note: always achievable by renumbering!)

completion $\hat{\mathcal{T}}$ consists then of all terms of the form

$$x_{i_j}^{\ell_{i_j}} x_{i_j+1}^{k_{i_j+1}} \cdots x_n^{k_n} \quad \text{with} \quad \forall m > i_j : k_m < \ell_m$$

(thus maximal degree of generator: $1 - r + \sum_{j=1}^{r} \ell_{i_j}$)

if "gap" exists at position m $\rightsquigarrow$ no bound for k_m

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Def: involutive division L *Noetherian* $\rightsquigarrow$ every finite $\mathcal{T} \subset \mathbb{T}(X)$ possesses involutive completion

Lemma: Janet division Noetherian

Proof: $s = \operatorname{lcm} \mathcal{T} \implies \hat{\mathcal{T}} = \{t \in \langle \mathcal{T} \rangle : t \mid s\}$ Janet basis of $\langle \mathcal{T} \rangle$

Remark: Pommaret division *not* Noetherian by example above

simplest counterexample: $\mathcal{T} = \{xy\} \subset \mathbb{T}(x, y)$

ideal $\langle \mathcal{T} \rangle$ contains *only* terms of class 1 $\rightsquigarrow$

infinite Pommaret “basis” $\{xy^k \mid k \in \mathbb{N}\}$

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Def: $\mathcal{I} \lhd \mathcal{P}$ polynomial ideal, finite set $\mathcal{H} \subset \mathcal{I}$ (term order $\prec$, involutive division L)

- $\mathcal{H}$ *weakly involutive basis* of $\mathcal{I}$ $\leadsto$ $\operatorname{lt}\mathcal{H}$ weakly involutive basis of $\operatorname{lt}\mathcal{I}$
- $\mathcal{H}$ *involutive basis* of $\mathcal{I}$ $\leadsto$ $\operatorname{lt}\mathcal{H}$ involutive basis of $\operatorname{lt}\mathcal{I}$ and all leading terms pairwise distinct

Lemma: $\mathcal{H}$ (weakly) involutive basis $\Longrightarrow$ $\mathcal{H}$ Gröbner basis

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Def: $\mathcal{I} \lhd \mathcal{P}$ polynomial ideal, finite set $\mathcal{H} \subset \mathcal{I}$ (term order $\prec$, involutive division L)

- $\mathcal{H}$ *weakly involutive basis* of $\mathcal{I}$ $\rightsquigarrow$ $\operatorname{lt}\mathcal{H}$ weakly involutive basis of $\operatorname{lt}\mathcal{I}$
- $\mathcal{H}$ *involutive basis* of $\mathcal{I}$ $\rightsquigarrow$ $\operatorname{lt}\mathcal{H}$ involutive basis of $\operatorname{lt}\mathcal{I}$ and all leading terms pairwise distinct

Lemma: $\mathcal{H}$ (weakly) involutive basis $\Longrightarrow$ $\mathcal{H}$ Gröbner basis

Prop: $\mathcal{H}$ *weakly* involutive basis $\Longrightarrow$ $\exists\, \mathcal{H}' \subseteq \mathcal{H}$ *strongly* involutive basis

Proof: as in monomial case

(weakly involutive bases required for generalisations like *semigroup orders* or polynomials over *rings* where stongly involutive bases generally do not exist)

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Def: $\mathcal{F} \subset \mathcal{P}$ finite set of polynomials

- *multiplicative variables* $\rightsquigarrow$ $\forall f \in \mathcal{F} : X_{L,\mathcal{F},\prec}(f) = X_{L,\operatorname{lt}\mathcal{F}}(\operatorname{lt} f)$
- *involutive span* $\rightsquigarrow$ $\langle \mathcal{F} \rangle_L = \sum_{f \in \mathcal{F}} \Bbbk[X_{L,\mathcal{F},\prec}(f)] \cdot f$

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Def: $\mathcal{F} \subset \mathcal{P}$ finite set of polynomials, $g \in \mathcal{P}$ further polynomial

- g *involutively head reducible* wrt $\mathcal{F}$ $\rightsquigarrow$ $\exists\, f \in \mathcal{F} : \operatorname{lt} f \,|_{L,\operatorname{lt}\mathcal{F}} \operatorname{lt} g$
- g *in involutive normal form* wrt $\mathcal{F}$ $\rightsquigarrow$ $\operatorname{supp} g \cap \langle \operatorname{lt} \mathcal{F} \rangle_L = \emptyset$
- $\mathcal{F}$ *involutively head autoreduced* $\rightsquigarrow$ $\nexists\, f_1 \neq f_2 \in \mathcal{F} : \operatorname{lt} f_1 \,|_{L,\operatorname{lt}\mathcal{F}} \operatorname{lt} f_2$

(*algorithms* for computing involutive normal forms or for involutive head autoreduction are obtained by trivial modifications of usual algorithms)

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Theorem: The following are equivalent:

(i) $\mathcal{H}$ weakly involutive basis of ideal $\mathcal{I} \lhd \mathcal{P}$

(ii) every $f \in \mathcal{I}$ possesses *involutive standard representation*

$$f = \sum_{h \in \mathcal{H}} P_h \cdot h \qquad \text{with} \quad \operatorname{lt}(P_h h) \preceq \operatorname{lt} f \wedge P_h \in \Bbbk[X_{L,\mathcal{H},\prec}(h)]$$

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$\mathcal{H}$ *strongly* involutive basis $\Longleftrightarrow$ involutive standard representation *unique*

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Proof:

- "$\Rightarrow$" compute involutive normal form
  "$\Leftarrow$" leading terms show that $\langle \operatorname{lt} \mathcal{H} \rangle_L = \operatorname{lt} \mathcal{I}$
- uniqueness follows from direct sum decomposition
  (at each step of normal form computation only *one* possible divisor!)

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Corollary:

- $\mathcal{H}$ weakly involutive basis $\implies$ $\langle\mathcal{H}\rangle_{L,\prec} = \mathcal{I}$
- $\mathcal{H}$ strongly involutive basis $\implies$ $\mathcal{I} = \bigoplus_{h\in\mathcal{H}} \Bbbk[X_{L,\mathcal{H},\prec}(h)] \cdot h$ ($\Bbbk$-linear direct sum decomposition)

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Caution: $\langle \mathcal{H} \rangle_{L,\prec} = \mathcal{I} \quad \not\Rightarrow \quad \mathcal{H}$ weakly involutive basis

Example: $\mathcal{H} = \{y^2, y^2 + x^2\}$ and $L = J$ (Janet division)

$\langle \mathcal{H} \rangle_{J,\prec} = \mathcal{I} = \langle \mathcal{H} \rangle$ *but* $x^2 \in \operatorname{lt} \mathcal{I} \setminus \langle \operatorname{lt} \mathcal{H} \rangle_J$

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Example: $\prec$ degree compatible order

$$\mathcal{F} = \{f_1 = z^2 - xy,\ f_2 = yz - x,\ f_3 = y^2 - z\} \subset \mathbb{k}[x, y, z]$$

- $\operatorname{lt} f_1 = z^2 \quad \Longrightarrow \quad \mathcal{F}$ Janet basis
- $\operatorname{lt} f_1 = xy \quad \Longrightarrow \quad f_4 = zf_1 + xf_2 = z^3 - x^2$ has *no* standard representation
  - $\mathcal{F} \cup \{f_4\}$ Gröbner basis, but *not* Janet basis
  - $\mathcal{F} \cup \{f_4,\ f_5 = zf_2\}$ Janet basis

Involutive bases are generally *non-reduced* Gröbner bases!

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Prop: $\mathcal{F} \subset \mathcal{P}$ finite, involutively head autoreduced set $\Longrightarrow$ involutive normal form of any polynomial $g \in \mathcal{P}$ wrt $\mathcal{F}$ *unique*

Proof: $\mathcal{F}$ induces direct sum decomposition of $\langle\mathcal{F}\rangle_{L,\prec}$ $\rightsquigarrow$ claim clear for $g \in \langle\mathcal{F}\rangle_{L,\prec}$: always involutive normal form 0 $\rightsquigarrow$

g_1, g_2 two different involutive normal forms of g $\Longrightarrow$
$g_1 - g_2 \in \langle\mathcal{F}\rangle_{L,\prec}$ in involutive normal form

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Prop: $\mathcal{F} \subset \mathcal{P}$ finite, weakly involutive set $\Longrightarrow$ *involutive* and *usual* normal form of any polynomial $g \in \mathcal{P}$ wrt $\mathcal{F}$ coincide

Proof:

- involutive normal form wrt weakly involutive basis unique (similar argument as above)
- usual normal form wrt Gröbner basis unique
- weakly involutive basis is Gröbner basis
- usual normal form trivially involutive normal form