Tutorial 1. General Involutive bases.

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Multiplicative indices

Exercise 1. Consider the following four sets of monomials in their respective polynomial rings. Compute the multiplicative indices of each of their elements using Thomas, Pommaret and Janet divisions.

1. $R = \mathbf{k}[x, y] \qquad T = \{x^2y, xy^2, y^3\}$
2. $R = \mathbf{k}[x, y, z] \qquad T = \{x^2z, xy^2z, xz^2, yz^2\}$
3. $R = \mathbf{k}[x, y, z] \qquad T = \{x^2yz, y^3, x^2y^2, y^2z^2, z^4\}$
4. $R = \mathbf{k}[x, y, z, t] \qquad T = \{x^2yz^2t, yt^3, xy^2z^2t, x^2y^3, yzt^3, x^4y, y^4\}$

   Fill the following table indicating in each cell whether the set is a (weakly) involutive basis with respect to each of the divisions:

	1	2	3	4
THOMAS				
JANET				
POMMARET				

This question is not easy to answer in general, we will see ways to answer it in the second lecture and tutorial.Can you think on an algorithmic way to do this for some division?

Exercise 2. Write CoCoA programs to compute multiplicative indices for the Thomas, Janet and Pommaret divisions. Two different versions are possible:

1. The program receives as input a set T of monomials (and an involutive division) and gives as output the set of multiplicative variables for each $t \in T$.

2. The program receives as input a set of T of monomials together with their respective multiplicative indices and a new monomial $m \notin T$ (and an involutive division). The program gives as output the set of multiplicative indices for each element of $T \cup \{m\}$.

Observe that once we have the second version, we can easily build the first one. A question is "which way to do it is more efficient?". The answer to this question might depend on the properties of the division. Consider for example what are differences of global and non-global division concerning this matter.

Exercise 3. As an application of the previous exercise, we can consider a combinatorial decomposition of the span of a set of monomials using the multiplicative indices with respect to some division. In case the given set is an involutive basis of the ideal generated by it, we have what is called a Stanley decomposition of the ideal. Write a CoCoA program that, given a set of monomials $\mathcal{S}$, gives the combinatorial decomposition based on the multiplicative indices of the elements $\mathcal{S}$.

Involutive divisions

Exercise 4. Check, using the definitions, that Janet and Pommaret divisions are actually involutive divisions.

Exercise 5. Write a CoCoA program that, given a weakly involutive set, produces a strong involutive set with the same involutive cone.

Exercise 6. Write a CoCoA program that, given a set of polynomials F with their multiplicative variables and a polynomial f gives the involutive normal form of f with respect to F. You can perform a slight modification of this algorithm to obtain the standard representation of f with respect to F.

Write a CoCoA program to compute involutive head autoreduction.

For CoCoALib/C++ programmers: All the programs in this and the subsequent tutorials can be programmed in CoCoALib instead of CoCoA4. For this, we have made available some classes and data types that can be used for the construction of the programs. We also provide an example program to see the use of these classes.