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Some Open Problems about Hilbert Functions

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Let $\mathbb{P}^n = \{[a_0 : a_1 : .... : a_n] \ \mid \ a_i \in k\}$.

$$P = k[x_0, \ldots, x_n] = \oplus_{j=0}^{\infty} P_j$$

$$I \subset P$$

will always be a homogeneous ideal.

Set $A = P/I$ then $A = \oplus_{j=0}^{\infty} A_j$ where $A_j = P_j/I_j$.

If $\mathbb{X} \subset \mathbb{P}^n$ then

$$I(\mathbb{X}) = \langle \{F \ \mid \ F \text{ homogeneous}, F(p) = 0, \text{for all } p \in \mathbb{X}\} \rangle$$

Example: Let $\mathbb{X} = \{p = [a_0 : a_1 : \ldots : a_n]\}$, i.e. a set consisting of a single point.

Then

$$I(\mathbb{X}) = (a_0x_1 - a_1x_0, a_0x_2 - a_2x_0, \ldots, a_0x_n - a_nx_0)$$

(i.e. n linearly independent linear equations).

So, $P/I(\mathbb{X}) \simeq k[T]$ and so the Hilbert function is always 1.

Now suppose that $\mathbb{X}$ is a set of s points, i.e. $\mathbb{X} = \{p_1, \ldots, p_s\}$. How can we find $I(\mathbb{X})_d$, i.e. the ideal of forms vanishing on $\mathbb{X}$ in degree d.

So, let $F \in P_d$, then $F = \sum a_\alpha x^\alpha$ where $\alpha = (\alpha_0, \ldots, \alpha_n)$, the $\alpha_i \geq 0$ and the sum of the α_i is equal to d.

Clearly $F(p_1) = \sum x^\alpha(p_1)a_\alpha$, i.e. this is a linear equation in the coefficients of F.

We get such a linear equation (in the a_α) for each point $p_1, \ldots, p_s$, i.e. we have a system of s homogeneous equations in $\binom{d+n}{n}$ unknowns.

The coefficient matrix of this system of homogeneous linear equations is an $s \times \binom{d+n}{n}$ matrix, M, whose i^{th} row consists of all the monomials of degree d evaluated at the point p_i.

The number of independent solutions of this system of equations will tell us the dimension of the ideal $I(\mathbb{X})_d$. I.e.

$$\dim(I(\mathbb{X})_d) = \binom{d+n}{n} - rank\ M.$$

Rewriting,

$$rank\ M = \binom{d+n}{n} - \dim(I(\mathbb{X})_d) = HF(P/I(\mathbb{X}), d)$$

In general, we expect this matrix to have rank as big as possible, i.e.

the expected Hilbert function for s points in $\mathbb{P}^n$ is $\min\{s, \binom{d+n}{n}\}$.

Theorem: A general set of s points in $\mathbb{P}^n$ has the expected Hilbert function.

(This is, in fact, an easy argument by induction on s.)

Now, points in $\mathbb{P}^n$ are the simplest example of linear subspaces of $\mathbb{P}^n$, i.e. subsets defined by linear homogeneous equations. So, a natural question is to ask what happens for families of general linear spaces of higher dimension!

This question is not simply a purely speculative question: answers to this and related questions about the Hilbert functions of general collections of linear spaces have applications in Algebraic Geometry (the dimensions of the secant varieties to various classical varieties) in Algebraic Statistics and in Communication Theory. It would take me too far afield to explain these connections. Fortunately, the questions are pretty enough on their own that I think that you won't need to have such motivation from outside of pure mathematics.

The first new case to consider is the case of lines in $\mathbb{P}^n$. For technical reasons it is best to consider this question first for families of linear spaces which have no pairwise intersections. So, for lines this means we should first consider the question for a general set of lines in $\mathbb{P}^n$ for $n \geq 3$.

If L is a line in $\mathbb{P}^n$, then $I(L) = (H_1, \ldots, H_{n-1})$, where the H_i are linearly independent linear forms. It follows that

$$P/I(L) \simeq k[T_1, T_2] \text{ and so } HF(P/I(L), t) = t + 1.$$

So, if X is a general set of s lines in $\mathbb{P}^n$ ($n \geq 3$) then the expected Hilbert function for $\mathbb{X}$ is:

$$HF(\mathbb{X}, t) = \min\left\{s(t+1), \binom{d+n}{n}\right\}.$$

Theorem: (Hartshorne, Hirschowitz)

A general set of s lines in $\mathbb{P}^n$ ($n \geq 3$) has the expected Hilbert function.

The proof of H-H is very interesting and the most difficult part of the proof is the case of lines in $\mathbb{P}^3$. The idea is to specialize the lines, but not so much, so that the special set of lines still has the "general" Hilbert function.

Corollary: Let $\mathbb{X}$ be a set of s general lines in $\mathbb{P}^n$ ($n \geq 3$) and s' general points, then

$$HF(X, t) \quad = \min\left\{s(t+1) + s', \quad \binom{d+n}{n}\right\}.$$

Next Step: **Planes, Lines and Points**

Since, in $\mathbb{P}^3$, a plane and a line always meet; and, in $\mathbb{P}^4$, two planes always meet, if we want to consider more than one plane, we should move into $\mathbb{P}^n$ for $n \geq 5$.

If, however, we only want to consider **One** plane, an arbitrary collection of general lines and an arbitrary collection of general points, we can start that study in $\mathbb{P}^4$.

First Expected Theorem: Consider the set $\mathbb{X}$ consisting of: one plane, s' general lines and s'' general points in $\mathbb{P}^n$ for $n \geq 4$. Then the Hilbert function of $\mathbb{X}$ should be:

$$HF(\mathbb{X}, t) \quad = \min\left\{ \binom{t+2}{2} + s'(t+1) + s'', \quad \binom{t+n}{n} \right\}$$

Theorem: The First Expected Theorem **IS** a Theorem! (joint work with E. Carlini, M.V. Catalisano).

Second Expected Theorem: Consider the set $\mathbb{X}$ consisting of s general planes, s' general lines and s'' general points in $\mathbb{P}^n$, $n \geq 5$. Then the Hilbert function of $\mathbb{X}$ should be:

$$HF(\mathbb{X}, t) = \min \left\{ s\binom{t+2}{2} + s'(t+1) + s'', \quad \binom{t+n}{n} \right\}.$$