June 11th, 2009

Warning: Sometimes during this tutorial we might have to stop some computations because they take too long or even be forced to restart the system.

Advice: Save any work you want to keep!

Exercise 1 *Let us assume we can use the initial ideals* $lt(I)$ *to compute the regularity of a polynomial ideal, or that our ideal is monomial from the very beginning. Consider the monomial ideal:*

$$I = \langle x_1, \ldots, x_{10} \rangle$$

1. *Compute it's regularity using* BettiDiagram.
2. *Do the same with* I^3 *(we already know the regularity of this ideal, this is just to check what kind of size this method can deal with).*

Even if the ideal is monomial we cannot compute the full resolution when our ideals start to grow. Of course, another alternative is to use $gin(I)$ *... what do you think?*

Exercise 2 *There is a lack of specific methods for monomial ideals, that take advantage of their combinatorial nature. In fact, it is an open problem to find combinatorial methods to compute the regularity of monomial ideals. We will play a bit with this problem in this tutorial.*

A first idea is to find (useful, important, big, etc..) families of monomial ideals such that we are able to compute their regularity. There are some: prime, stable, squarefree stable, some ideals corresponding to fat points, some edge ideals of graphs, generalized k*-out-of*n *ideals, etc ...*

What about ideals of nested type *it would be great (remember tutorial 3) to have a method for them. In fact there are several [cf. Bermejo & Gimenez 2006]:*

Proposition 1 *Let $I \subset R$ be a monomial ideal of nested type. If d is the dimension of R/I and p is the least integer such that none of the minimal generators of I involves $x_{p+1}, \ldots, x_n$ then*

$$\begin{aligned} reg(I) \;= \max\{ \; & \operatorname{sat}(I \cap \mathbf{k}[x_0, \ldots, x_p]), \\ & \operatorname{sat}(I|_{x_p=1} \cap \mathbf{k}[x_0, \ldots, x_{p-1}]), \\ & \operatorname{sat}(I|_{x_{p-1}=1} \cap \mathbf{k}[x_0, \ldots, x_{p-2}]), \\ & \cdots \cdots \cdots \\ & \operatorname{sat}(I|_{x_{n-d+1}=1} \cap \mathbf{k}[x_0, \ldots, x_{n-d}]) \; \}. \end{aligned}$$

Proposition 2 *Let $I \subset R$ be a monomial ideal of nested type. Let $x_0^{\lambda_0} \ldots x_n^{\lambda_n}$ be the least common multiple of its minimal generators. For all $i \in \{0, \ldots, n\}$ let δ_i be the least degree of the minimal generators of $I^\star := (x_0^{\lambda_0+1}, \ldots, x_n^{\lambda_n+1}) : I$ involving exactly the variables $x_0, \ldots, x_i$, if any. Otherwise, set $d_i := 0$. Then,*

$$reg(I) = \max_{n-dimR/I \leq i \leq n-depth(R/I)} \{\lambda_0 + \cdots + \lambda_i + 1 - \delta_i; \delta_i \neq 0\}$$

Proposition 3 *Let $I \subset R$ be a monomial ideal of nested type. Let $I = \mathfrak{q}_1 \cap \cdots \mathfrak{q}_r$ be the irredundant irreducible decomposition of I. Then*

$$\operatorname{reg}(I) = max\{reg(\mathfrak{q}_i); 1 \leq i \leq r\}$$

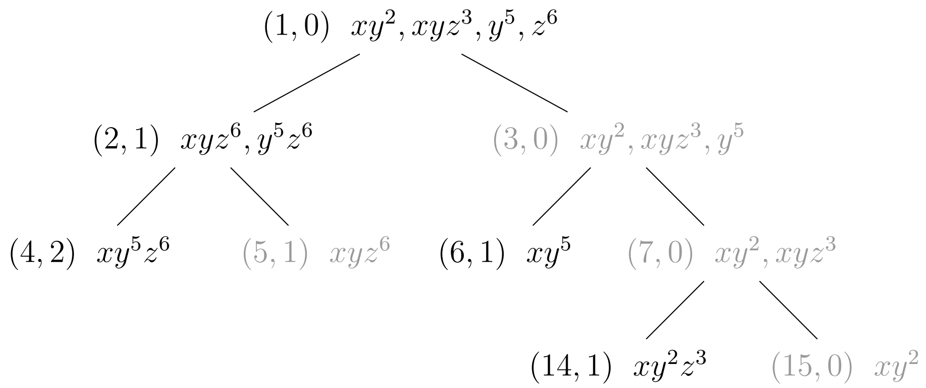
Choose one of these propositions and implement a method to compute the regularity of a monomial ideal of nested type. Proposition 1 is included for the sake of completeness or in case you feel better using it. we recomend to use Proposition 2 or Proposition 3. What is the regularity of an irreducible monomial ideal? If you are interested in efficiency, ask for efficient methods to compute irreducible decompositions of monomial ideals in CoCoALib.

You can use your implementation to complete the method of tutorial 3 to compute the regularity of any polynomial ideal.

Exercise 3 *What if our monomial ideal is not of nested type or does not belong to one of the families for which we know how to compute regularity? Is there any hope to compute their regularity in a reasonable time?. Any way to obtain good bounds at least?*

There is a procedure to compute the ranks of a multigraded resolution without computing the resolution itself (i.e. without computing the differentials). Therefore we obtain bounds for the Betti numbers of the ideal (in particular, for the regularity), dwelling a bit in this method we can find ways to compute the regularity of general monomial ideals (or at least good bounds in the worst case) in a reasonable time even for big ideals.

Let $I = \langle m_1, \ldots, m_r \rangle$ a monomial ideal. Denote $I' := \langle m_1, \ldots, m_{r-1} \rangle$ and $\tilde{I} = I' \cap \langle m_r \rangle$. Observe that $\tilde{I}$ is generated by $\{lcm(m_i, m_r); 1 \leq i \leq r-1\}$. Let us construct a tree in which each node contains a monomial ideal, and is labeled by a position and a dimension. The root of this tree contains I and has position 1 and dimension 0. Given a node (J, p, d) in the tree it has two children: $(J', 2p+1, d)$ on the right and $(\tilde{J}, 2p, d)$ on the left. This is what we call a Mayer-Vietoris tree *of I, denoted $MVT(I)$. We say that the root and the nodes in even position are called* relevant nodes. *The relevant nodes support a resolution of I, and the multidegrees of the modules in homological degree i of this resolution are those generators of the relevant nodes of dimension i in $MVT(I)$.*



Proposition 4 *Let I be a monomial ideal and $MVT(I)$ a Mayer-Vietoris tree of I. Let us define the following numbers for every i and μ:*

$$\bar{\beta}_{i,\mu}(I) = \begin{cases} 1 & \mu \text{ appears only once as a generator of a relevant node in } MVT(I) \\ 0 & \text{in any other case} \end{cases}$$

$$\tilde{\beta}_{i,\mu}(I) = \#\{ \text{ Times } \mu \text{ appears as a generator of a relevant node in } MVT(I)\}$$

Then

$$\bar{\beta}_{i,\mu}(I) \leq \beta_{i,\mu}(I) \leq \tilde{\beta}_{i,\mu}(I) \quad \forall i, \mu$$

Therefore, using Mayer-Vietoris trees and this proposition we have upper and lower bounds for the regularity of a monomial ideal. Of course, when these bounds coincide, we obtain the actual regularity.

We have implemented a CoCoALib *version of the Mayer-Vietoris tree algorithm that can be used to obtain bounds for the regularity of a monomial ideal. Implement a procedure in* CoCoA *to produce random ideals and run the Mayer-Vietoris function to see whether it gives you the actual regularity or just bounds (and how tight these bounds are).*