June 10th, 2009

Warning: Sometimes during this tutorial we might have to stop some computations because they take too long or even be forced to restart the system.

Advice: Save any work you want to keep!

Exercise 1 *Is not easy to compute the regularity of a polynomial ideal I. One needs either to compute the Betti numbers of I or $gin(I)$. Both are hard computations. It would be easier if we could at least use the initial ideal $lt(I)$. Consider the homogeneous ideal:*

$$I = \langle x^2 - yz + 3tu, xyz^2 + z^4, xyt - 3u^3, x^2t^2 + 4y^2u^2 \rangle$$

1. *Compute it's regularity using* BettiDiagram
2. *Compare it's regularity and Betti numbers with those of* LT(I)
3. *Compute it's regularity using* Gin *and* Gin5
4. *Do the same with I^3*
5. *What about I^4 ?*
6. *Let now $I = \langle x_1, \ldots, x_{20} \rangle$. Compute* Gin(I) *and* BettiDiagram(I). *What is the regularity of this ideal?*

Exercise 2 *$reg(lt(I))$ gives in general an upper bound for $reg(I)$. But sometimes both regularities coincide. In particular, if $lt(I)$ is* of nested type *we have that $reg(I) = reg(lt(I))$.*

Definition 1 *A monomial ideal $I \subset R$ is said to be* of nested type *(Bermejo & Gimenez 2006) if, for any prime ideal $\mathfrak{p} \subset R$ associated to I, there exists $i \in \{0, \ldots, n\}$ such that $\mathfrak{p} = (x_0, \ldots, x_i)$.*

Proposition 1 *Let $I \subset R$ be a monomial ideal and set $d := dim R/I$. The following conditions are equivalent:*

1. *I is of nested type*

2. $\forall i \in \{0, \dots, n\}$, $I : (x_i)^\infty = I : (x_0, \dots, x_i)^\infty$.

3. *x_n is not a zero divisor on R/I^{sat}, and for all $i : n-d+1 \le i < n$, x_i is not a zero divisor on $R/(I, x_n, \dots, x_{i+1})^{sat}$.*

4. (a) $\forall i \in \{0, \dots, n-d\}$, *there exists* $k_i \ge 1$ *such that* $x_i^{k_i} \in I$, *and*

   (b) $I : (x_n)^\infty \subseteq I : (x_{n-1})^\infty \subseteq \cdots \subseteq I : (x_{n-d+1})^\infty$.

Implement a procedure to detect whether a monomial ideal is of nested type.

Exercise 3 *If our initial ideal is not of nested type a good strategy (see Bermejo & Gimenez 2006) consists of associating to I an ideal of nested type $N(I)$ such that $reg(I) = reg(N(I))$. The way to find $N(I)$ is via a coordinate transformation in I to obtain I' and then $N(I) = lt(I')$. We want to do "smaller" coordinate transformations than those needed to compute $gin(I)$.*

Let $d := dim R/I$. There are two cases:

1. *If $lt(I)$ is not of nested type and $\mathbf{k}[x_n - d + 1, \dots, x_n]$ is a Noether normalization of R/I then consider the following transformation:*

$$\begin{array}{rcl} x_n & \mapsto & x_n + t_1 x_{n-1} + t_2 x_{n-2} + \cdots + t_{d-1} x_{n-d+1} \\ x_{n-1} & \mapsto & x_{n-1} + t_d x_{n-2} + \cdots + t_{2d-3} x_{n-d+1} \\ \vdots & & \\ x_{n-d+2} & \mapsto & x_{n-d+2} + t_{\frac{d(d-1)}{2}} x_{n-d+1} \end{array}$$

2. *If $lt(I)$ is not of nested type and $\mathbf{k}[x_n - d + 1, \dots, x_n]$ is not a Noether normalization of R/I then we need a different (bigger) transformation:*

$$\begin{array}{rcl} x_n & \mapsto & x_n + t_1 x_{n-1} + t_2 x_{n-2} + \cdots + t_n x_0 \\ x_{n-1} & \mapsto & x_{n-1} + t_{n+1} x_{n-2} + \cdots + t_{2n-1} x_0 \\ \vdots & & \\ x_{n-d+1} & \mapsto & x_{n-d+1} + \cdots + t_{dn - \frac{d(d-1)}{2}} x_0 \end{array}$$

With these coordinate changes we obtain generically *the ideal $N(I)$ we are looking for. Therefore our procedure will follow the scheme:*

1. *Decide whether $\mathbf{k}[x_n - d + 1, \dots, x_n]$ is a Noether normalization.*

2. *Apply a random change of coordiantes of the required form depending on the response at 1) obtaining I'.*

3. *Check wether $lt(I')$ is of nested type. If not, apply another random change of coordinates until $lt(I')$ is of nested type. (This procedures always terminates when the base field has characteristic 0.)*

To Do:

• *Implement a procedure to detect whether $\mathbf{k}[x_n - d + 1, \ldots, x_n]$ is a Noether normalization.*

Hint: *You can use the following result [Bermejo&Gimenez, 2001]*

Lemma 1 *Let I be a homogeneous ideal of $S = \mathbf{k}[x_0, ..., x_n]$ such that $d = dim(S/I) - 1$ and denote by $in(I)$ the initial ideal of I w.r.t. the reverse lexicographic order. The following are equivalent:*

1. $\mathbf{k}[x_{n-d}, ..., x_n]$ *is a Noether normalization of S/I*

2. $\forall i : 0 \leq i \leq n - d - 1$, *there exists $r_i \in \mathbb{N} - 0$ such that $x_i^{r_i} \in in(I)$*

3. $dim(S/(I, x_{n-d}, ..., x_n)) = 0$

4. $dim(S/(in(I), x_{n-d}, ..., x_n)) = 0$

• *Implement the coordinate changes.*

• *Implement the full procedure, i.e. given a polynomial ideal I, obtain a monomial ideal of nested type $N(I)$ such that $reg(I) = reg(N(I))$.*