CoCoA School 7-12 June 2009
Castelnuovo-Mumford regularity and applications

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Contents

Reg-limited

The Krull dimension d, the multiplicity e and the Castelnuovo Mumford regularity are important invariants and it is an interesting problem to study the possible relations among them.

- If $A = P/I$ is Cohen-Macaulay, we have seen that $reg(A) =$degree of the h-polynomial $h_A(z)$. Hence (we may assume $I \subseteq m^2$)

  $$reg(A) \leq e - n + d$$

  In particular $=$ holds $\iff$ $h_A(z) = 1 + (n-d)z + \cdots + z^{e-n+d-1}$.

- The following example shows that in general the regularity cannot be bounded by a function $F(e, d, n)$.

  Example. Let $r \in \mathbb{N}^*$ and consider $A = k[x, y]/(x^2, xy^r)$ which is 1-dimensional non C-M. In this case $e(A) = 1$ but $reg(A) = r$.

  Geometric information can produce better situations.

Reg-limited

- (Castelnuovo) $I = I(C)$ where C is a smooth curve:

$$reg(I) \leq e - 1$$

More in general

- (Gruson-Lazarsfeld-Peskine) $k = \overline{k}$, $I = I(C)$ where C is a reduced irreducible curve in $\mathbb{P}^n$:

$$reg(I) \leq e - n + 2$$

The regularity has been used by S. Kleiman in the construction of bounded families of sheaves with given Hilbert polynomial, a crucial point in the construction of Hilbert or Picard scheme.

Here we will present a result by Kleiman in the case of equidimensional reduced schemes.

The problem is related to the finitness of Hilbert functions for classes of graded algebras with given multiplicity.

Reg-limited

Let $\mathcal{C}$ be a class of homogeneous ideals in P, then we say:

- $\mathcal{C}$ **is HF-finite** if the number of numerical functions which arise as the Hilbert functions of P/I, $I \in \mathcal{C}$, is finite,
- $\mathcal{C}$ is **HP-finite** if the number of polynomials which arise as the Hilbert polynomials of P/I, $I \in \mathcal{C}$, is finite,
- $\mathcal{C}$ is **reg-limited** if for some integer t and all $I \in \mathcal{C}$ we have $reg(P/I) \leq t$,
- $\mathcal{C}$ is **g-reg-limited** if for some integer t and all $I \in \mathcal{C}$ we have $g\text{-}reg(P/I) \leq t$

  ($g\text{-}reg(P/I) = reg(P/I^{sat})$ called the geometric regularity).

Reg-limited

Fix $P = k[x_1, \ldots, x_n]$ and let $\mathcal{C}$ be a class of homogeneous ideals in P

$$\mathcal{C} \text{ reg-limited} \iff \mathcal{C} \text{ HF-finite}$$

- $(\Longrightarrow)$ Assume

$$t \geq reg(P/I) = reg(P/gin_{revlex}(I)) \geq m - 1$$

  where $m =$ maximum degree of the generators of $gin(I)$. Since $HF_{P/I}(n) = HF_{P/gin_{revlex}(I)}(n)$ and the monomials of degree $\leq t + 1$ in P are a finite number, the conclusion follows

- $(\Longleftarrow)$ For the converse, since $HF_{P/I}(n) = HF_{P/Lex(I)}(n)$, if $\mathcal{C}$ is HF-finite, there are only a finite number of lexicographic ideals in P associated to $\mathcal{C}$. Now the result follows because

$$reg(P/I) \leq reg(P/Lex(I)).$$

g-reg-limited

We have seen (example) that

$$\mathcal{C} \quad \text{HP-finite} \quad \not\Longrightarrow \mathcal{C} \quad \text{reg-limited}$$

If $\mathcal{C}$ is HP-finite, then we have a uniform upper bound for the geometric regularity of P/I in $\mathcal{C}$.

$$\mathcal{C} \quad \text{HP-finite} \quad \Longrightarrow \mathcal{C} \quad \text{g-reg-limited}$$

It is a consequence of Gotzmann's formula which says:
the Hilbert polynomial of $A = P/I$ can be written in the unique form

$$HP_A(X) = \binom{X + a_1}{a_1} + \binom{X + a_2 - 1}{a_2} + \cdots + \binom{X + a_s - (s-1)}{a_s}$$

with $a_1 \geq a_2 \geq \cdots \geq a_s \geq 0$, then

$$reg(P/I^{sat}) \leq s - 1.$$

Kleiman's theorem

For every $d \geq 1$ we define recursively the following polynomials $F_d(X)$ with rational coefficients. We let

$$F_1(X) = X - 1, \quad F_2(X) = X^2 + X - 1$$

and if $d \geq 3$ then we let

$$F_d(X) = F_{d-1}(X) + X\binom{F_{d-1}(X) + d - 1}{d - 1}.$$

Theorem

Let $A = P/I$ be a reduced equidimensional graded algebra of dimension d and multiplicity e. Then

$$reg(A) \leq F_d(e).$$

We can list the main steps of an algebraic proof (by Rossi, Trung and Valla).

Kleiman's theorem: an algebraic proof

Throughout this part we need the assumption that the field K is algebraically closed and of characteristic zero. We need it in order to use Bertini-type theorem on the generic hyperplane section of a reduced and non degenerate variety (see Flenner's result).

- Since we want to use induction on the dimension $d \geq 2$ of $A = P/I$, we will choose a generic element $z \in P_1$ and we consider

  $$B := P/(I + zP)^{sat}.$$

  It is clear that $\dim(B) = d - 1$ and

  $$e(A) = e(A/zA) = e(P/(I + zP)) = e(B) := e.$$

  If we assume that A is reduced and equidimensional, then B is reduced equidimensional too (Flenner's result).
- Hence we need to control $g\text{-}reg(A/zA)$ where z is a general hyperplane section.

Mumford's theorem: an algebraic approach

- Unlike the regularity, the geometric regularity does not behave well under generic (and regular) hyperplane sections. Take for example the standard graded algebras

$$A = k[X, Y, Z]/(X^2, XY), \quad T = k[X, Y]/(X^2, XY).$$

Then $g\text{-}reg(A) = reg(A) = 1$ while $g\text{-}reg(T) = 0,\ reg(T) = 1.$

However the following crucial result gives us the opportunity to control this bad behaviour.

Theorem (An algebraic version of Mumford's theorem)

Let $A = P/I$ be a standard graded algebra and $z \in A_1$ a regular linear form in A. If $g\text{-}reg(A/zA) \leq m$, then

$$reg(A) \leq m + \dim(H^1(A)_m) = m + HP_A(m) - HF_A(m)$$

HF-finite

Hence, since $HP_A(m)$ can be bounded in terms of the multiplicity and the dimension and m can be controlled by induction, we can prove Kleiman's result.

Corollary

Let $\mathcal{C}$ be the class of reduced equidimensional graded algebras with given multiplicity and dimension. Then $\mathcal{C}$ is HF-finite.

We need only to remark that if $P = k[X_1, \dots, X_n]$ and $A = P/I$ has dimension d and multiplicity e, then $n - d + 1 \leq e$. The conclusion follows by Kleiman's theorem.

The theorem does not hold even if we consider reduced graded algebras not necessarily equidimensional. Take for example the graded rings

$$A_r := k[X, Y, Z, T, W]/(X) \cap (W, XZ^r - YT^r).$$

All the elements of the family have dimension four, molteplicity one, but the regularity and the Hilbert function depend on r.

HF-finite

Kleiman's theorem does not hold if we delete the assumption that every element of the family is reduced.

Take for example the graded rings

$$A_r := k[X, Y, Z, T]/(Y^2, XY, X^2, XZ^r - YT^r).$$

This is the coordinate ring of a curve in $\mathbf{P}^3$ which can be described as the divisor $2L$ (L is a line) on a smooth surface of degree $r+1$. The Hilbert series of A_r is

$$HS_{A_r}(z) = \frac{1 + 2z - z^{r+1}}{(1-z)^2}$$

so that

$$\dim(A_r) = 2, \quad e(A_r) = 2$$

but $reg(A_r) = r$ and we do not have a finite number of Hilbert functions.