CoCoA School 7-12 June 2009
Castelnuovo-Mumford regularity and applications

Maria Evelina Rossi

Università di Genova
Dipartimento di Matematica

TUTORS COCOA: Anna Bigatti and Eduardo De Cabezon Irigaray

Contents

1. Hilbert Functions and graded minimal free resolutions
2. Castelnuovo Mumford Regularity and its behavior relative to Hyperplane sections, Sums, Products, Intersections of ideals
3. Castelnuovo Mumford regularity and initial ideals
4. Finiteness of Hilbert Functions and regularity
5. Bounds on the regularity and Open Problems

References

Term ordering

$P = k[x_1, \dots, x_n]$
$\mathbb{T}^n = \{\text{terms of } P\} = \{m = x_1^{\alpha_1} \cdots x_n^{\alpha_n} : (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$

A term ordering (or *term order or monomial ordering) is a total order τ such that*

- $m >_\tau 1$ *for every non constant term*
- $m_1 >_\tau m_2$ *and* $m_3 \in \mathbb{T}^n$ *then* $m_1 m_3 >_\tau m_2 m_3$

Revlex order

$m = x_1^{a_1} \cdots x_n^{a_n} >_{revlex} n = x_1^{b_1} \cdots x_n^{b_n} \iff$ *the LAST non-zero entry of the vector* $(a_1 - b_1, \dots, a_n - b_n)$ *is NEGATIVE.*

For example:
$x_2^2 >_{revlex} x_1 x_3$ *because* $(a_1 - b_1, \dots, a_n - b_n) = (-1, 2, -1)$

Invariants and deformations

Let τ a term order $\longrightarrow$ initial ideal $Lt_\tau(I)$

- I and $Lt_\tau(I)$ have the same Hilbert function
- $\beta_{ij}(I) \leq \beta_{ij}(Lt_\tau(I))$

  As a consequence

- $pd_P(I) \leq pd_P(Lt_\tau(I))$
- $reg(I) \leq reg(Lt_\tau(I))$

(usually <) Check with CoCoA in some examples.

Properties of $\tau =$ RevLex

Let I be an homogeneous ideal in $P = k[x_1, \ldots, x_n]$, $\tau =$ RevLex

- $F \in P$ homogeneous

$$Lt_\tau(F) \in (x_s, \ldots, x_n),\ 1 \leq s \leq n \implies F \in (x_s, \ldots, x_n)$$

- $Lt_\tau(I + (x_n)) = Lt_\tau(I) + (x_n)$
- $Lt_\tau(I : x_n) = Lt_\tau(I) : x_n$
- $x_n, \ldots, x_s$ is a P/I-regular sequence $\iff$ $x_n, \ldots, x_s$ is a $P/Lt_\tau(I)$-regular sequence

Gin(I): the generic initial ideal

Let $g : P \to P$ be a graded automorphism of k-algebras defined by $x_i \rightsquigarrow L_i$, with L_i linear forms

$$g \quad \longleftrightarrow M \in GL_n(k)$$

given by the coefficients of L_i w.r.t. x_i. In particular $GL_n(k)$ acts on P by

$$g(x_i) = \sum_{j=1}^{n} g_{ji} x_j$$

$$g(F(x_1, \dots, x_n)) = F(g(x_1), \dots, g(x_n))$$

It is clear that $P/I \simeq P/g(I)$.

Given τ, I an ideal, we consider the family $Lt_\tau(g(I)), \quad g \in GL_n(k)$

Gin(I): the generic initial ideal

For a generic $g \in GL_n(K)$, $Lt_\tau(g(I))$ is *constant:*

Theorem (Galligo, Bayer-Stilmann)

There exists $U \neq \emptyset$ a Zariski-open subset of $GL_n(k)$ such that

$$Lt_\tau(g(I)) = Lt_\tau(h(I))$$

for every $g, h \in U$.

Set

$$gin_\tau(I) := LT_\tau(g(I)) \quad \textit{for every } g \in U$$

Bayer-Stilman's Theorem

Theorem (Bayer-Stilmann)

Let $I \subseteq P$ be an homogeneous ideal, $|k| = \infty$, $\tau =$ revlex.

$$reg(P/I) = reg(P/gin_\tau(I))$$

Proof: We recall that $gin_\tau(I) = Lt_\tau(g(I))$. The result is clear if $d = dimP/I = 0$. Assume $d > 0$. Since $\tau =$revlex we know that

$$gin_\tau(I : x_n) = gin_\tau(I) : x_n \quad [\text{ and } \quad gin_\tau(I + (x_n)) = gin_\tau(I) + (x_n)].$$

As a consequence $I : x_n/I$ and $gin_\tau(I) : x_n/gin_\tau(I)$ have the same HF. We may assume that they have finite length (after a general linear change of coordinates), hence they also have the same regularity. Moreover $reg(P/I + (x_n)) = reg(P/gin_\tau(I) + (x_n))(= reg(P/gin_\tau(I + (x_n))))$ by induction. Then

$$reg(P/I) = \max\{reg(I : x_n/I), reg(P/I + (x_n))\} =$$

$$= \max\{reg(gin_\tau(I) : x_n/gin_\tau(I)), reg(P/gin_\tau(I) + (x_n))\} = reg(P/gin_\tau(I)).$$

Bayer-Stilman's Theorem in char $k = 0$

Theorem (Bayer-Stilmann)

Let $I \subseteq P$ be an homogeneous ideal, char $k = 0$ $\tau =$ revlex.

$$reg(P/I) = reg(P/gin_\tau(I)) = \text{ max degree of a generator of } gin_\tau(I)$$

It can be deduced from the fact :

char $k = 0 \quad \Longrightarrow \quad gin_\tau(I)$ strongly stable monomial ideal

(i.e. for any monomial M, $x_i M \in J \Longrightarrow x_j M \in J, \forall j \leq i$)

By Eliahou-Kervaire's resolution of stable ideals J

$reg\ J =$ maximum degree of generator.

A refinement of Bayer-Stilman's Theorem

Definition

A Betti number $\beta_{ij}(I)$ is extremal if

$$\beta_{rs}(I) = 0 \quad \forall r > i \quad \text{and} \quad s > j$$

In the Betti Diagram it means that β_{ij} extremal is in the left corner

Theorem (Bayer-Charalambous-Popescu)

Let $I \subseteq P$ be an homogeneous ideal, $\tau =$ revlex. If $\beta_{ij}(I)$ is extremal then

$$\beta_{ij}(I) = \beta_{ij}(gin_\tau(I))$$

By using results by P. Schenzel (see Barcelona '96) we can translate the extremality of the Betti numbers into an extremality of the graded pieces of the local cohomology modules.

Exercise

Exercise. Given $I = (x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2^2, x_3x_5) \subseteq P = k[x_1, \dots, x_5]$.
Compute

1) gin-revlex
2) gin-lex
3) lexicographic ideal

and compare the Castelnuovo regularities.

Bermejo-Gimenez, Trung result

Recent papers by G. Caviglia-E. Sbarra, G. Caviglia, Bermejo- Gimenez and Trung show that the regularity may also be computed by replacing $gin_{revlex}(I)$ with $Lt_{revlex}(I)$ provided that $Lt_{revlex}(I)$ has only associated prime ideals of type $(x_1, x_2 \ldots, x_k)$. This can be verified on the initial ideal via some *genericity conditions* we are going to explore via the filter regular elements.

Definition

An element $x \in P_1$ is filter regular for P/I if

$$(P/I)_i \xrightarrow{\cdot x} (P/I)_{i+1}$$

is injective for $i \gg 0$.
Equivalently $x \notin \wp \ \forall \wp \in Ass(I), \ \wp \neq m$.

Hence x is filter regular iff

$$(I : x)_i = I_i \quad \forall i \gg 0 \ \text{ or } \ \text{equivalently} \quad \lambda(I : x/I) < \infty$$

Bermejo-Gimenez, Trung method: basic idea

Remark If x is filter regular for P/I, then

$$reg(P/I) = \max\{reg(I : x_n/I), reg(P/I + (x_n))\}$$

Definition

$y_1, \dots, y_t$ is a filter regular sequence for P/I if y_1 is filter regular and y_i is filter regular in $P/(y_1, \dots, y_{i-1})$ for every $i = 2, \dots, t$.

- Let $y_1, \dots, y_t$ be a filter regular sequence for P/I. Then $y_1, \dots, y_t$ is a s.o.p. in P/I
- If $|k| = \infty$ then there exists a maximal filter regular sequence $y_1, \dots, y_d$ where $d = \dim P/I$.
- $[I + (y_1, \dots, y_i) : y_{i+1}]_r = [I + (y_1, \dots, y_i)]_r \ \forall r \gg 0.$

Bermejo-Gimenez, Trung's result

Assume $d \geq 1$. Let $\underline{y} := y_1, \ldots, y_d$ be a sequence of linear forms. Define

$$I_i := I_{i-1} + (y_i) \quad (I_0 = I)$$

$$a_{\underline{y}}^i(P/I) := reg(I_{i-1} : y_i/I_{i-1}) = \sup\{r : \ [I_{i-1} : y_i]_r \neq [I_{i-1}]_r\}$$

with $a_{\underline{y}}^i = -\infty$ if $I_{i-1} : y_i = I_{i-1}$.

$\underline{y} := y_1, \ldots, y_d$ is a filter-regular sequence for P/I if and only if $a_{\underline{y}}^i < \infty$.

We control the regularity in terms of these integers:

Theorem

Let $\underline{y} := y_1, \ldots, y_d$ be a maximal filter regular sequence for P/I. Then

$$reg(P/I) = \max\{a_{\underline{y}}^i(P/I) : \ 1 \leq i \leq d; reg(P/I_d)\}$$

Bermejo-Gimenez, Trung's algorithm

Let $\underline{x} := x_n, \ldots, x_{n-d+1}$, by the properties of τ=revlex we have

$$a^i_{\underline{x}}(P/I) = a^i_{\underline{x}}(P/Lt_{revlex}(I))$$

Theorem

Let $\underline{x} := x_n, \ldots, x_{n-d+1}$. If $a^i_{\underline{x}}(P/Lt_{revlex}(I)) < \infty$ then

$$reg(P/I) = reg(P/Lt_{revlex}(I))$$

Bermejo-Gimenez, Trung's algorithm

- Consider a sufficiently general (*not so much*) change of coordinates
- Compute $a^i_{\underline{x}}(P/Lt_{revlex}(I))$ where $\underline{x} := x_n, \ldots, x_{n-d+1}$
- The generality is enough if $\underline{x} := x_n, \ldots, x_{n-d+1}$ is a filter regular sequence for $P/Lt_{revlex}(I))$ (equivalently for P/I), that is $a^i_{\underline{x}}(P/Lt_{revlex}(I)) < \infty$
- If $a^i_{\underline{x}}(P/Lt_{revlex}(I)) < \infty$ then

$$reg(P/I) = reg(P/Lt_{revlex}(I))$$

Problem: Implement a CocoA function BGT-reg(I).