# The Segre Varieties

I would like to now turn to the family of Segre Varieties and see what the state of the art is there. The investigation of the Secant Varieties of the Segre Varieties is a very active area of current research because many applications outside of algebraic geometry can be found for solutions to the problems here.

As this is a very active area of research, it is still not completely clear what is to be expected, but a great deal of data is being collected about what is true and what is not true and that is helping to make it possible to begin to formulate conjectures.

Exactly what is the problem: Let's consider $t \geq 3$ vector spaces $V_1, \ldots, V_t$, where $\dim V_i = n_i + 1$. Let's suppose also that $n_1 \leq n_2 \leq \cdots \leq n_t$. Let

$$V = V_1 \otimes \cdots \otimes V_t.$$

We say that $v \in V$ is *decomposable* if $v = v_1 \otimes \cdots \otimes v_t$ for $v_i \in V_i$.

After choosing bases for each $V_i$ we get a basis for $V$ of decomposable vectors choosing a vector from the basis for each $V_i$. Thus, $\dim V = \Pi_{i=1}^{t} \dim V_i$. This basis consists of decomposable vectors.

After having chosen bases for the $V_i$, and using them to get a basis for $V$, we can arrange the coordinates of vectors of $V$ in a sort of $t$-dimensional array (for $t = 2$, this was a matrix, for $t = 3$ it looks like a box, etc. ).

**Definition:** A vector $v \in V$ has *tensor rank* $r$ if

$$v = T_1 + \cdots + T_r$$

where $T_i$ is a decomposable tensor, and no shorter such decomposition exists.

**Note:** 1) Since $v$ and $\lambda v$, $\lambda \neq 0$ clearly have the same tensor rank, the question of tensor rank can be posed in $\mathbb{P}(V)$.

2) Since $V$ has a basis of decomposable vectors, every vector $v \in V$ has finite tensor rank.

Several problems emerge, some from the applications anticipated.

*i*) What is the maximum tensor rank for $v \in V$?

*ii*) How do you calculate the tensor rank of a given $v \in V$?

*iii*) What is the "generic" tensor rank, i.e. if

$$U_\ell = \{v \in V \mid \text{ the tensor rank of } v \leq \ell\}$$

what is the least $\ell$ for which $\overline{U_\ell} = \mathbb{P}(V)$?

This least $\ell$ is called the *tensor rank of* $V$.

We've seen that for $t = 2$, all of these questions have relatively simple answers (explain).

Let me give you the first indication of how things go badly for $t > 2$. I will construct an example (for $t = 3$) in which $\overline{U_\ell} = \mathbb{P}(V)$ but there are tensors of tensor rank strictly bigger than $\ell$ in $V$.

Let's look at the case of three vector spaces, all of dimension 2, call them $V_1, V_2, V_3$. I'll choose bases for each and look at the elements of $V = V_1 \otimes V_2 \otimes V_3$ as $2 \times 2 \times 2$ boxes.

One can speak of the "faces" of a box. It is an easy exercise to see that "elementary face operations" don't change the tensor rank.

I will now introduce some notation for these $2 \times 2 \times 2$ boxes:

$$B = \left\{ \begin{matrix} M_1 \\ \\ M_2 \end{matrix} \right\}$$

where $M_1$ denotes the top face and $M_2$ the bottom face.

Clearly

$$B = \left\{ \begin{matrix} M_1 \\ \\ 0 \end{matrix} \right\} + \left\{ \begin{matrix} 0 \\ \\ M_2 \end{matrix} \right\}$$

and each summand is a sum of two decomposable vectors. The first is

$$(1,0) \otimes v_2 \otimes v_3 + (1,0) \otimes v_2' \otimes v_3'$$

and the second is

$$(0,1)\otimes w_2\otimes w_3+(0,1)\otimes w_2'\otimes w_3'$$

(explain how the places in the box are determined.)

It follows from what we did for matrices that every $2\times 2\times 2$ tensor is a sum of $\leq 4$ decomposable vectors. But, notice that if one of these faces was a matrix of rank 1, we would have gotten away with three summands.

I want to show that every $2\times 2\times 2$ tensor is a sum of $\leq 3$ decomposable tensors. So, it is enough to consider the case in which every face has rank $=2$.

So, suppose we have at least one "ray" in the box where we have $(x\ \ y)$ with $xy\neq 0$. Suppose the appropriate faces are

$$M_1=\begin{pmatrix} x & b\\ c & d\end{pmatrix}, M_2=\begin{pmatrix} y & f\\ g & h\end{pmatrix}$$

Let

$$B_1=\begin{Bmatrix} N_1\\ \\ 0\end{Bmatrix} \text{ where } N_1=\begin{pmatrix} x & b\\ c & \frac{bc}{x}\end{pmatrix}$$

Notice that $rk\ N_1 = 1$ and so $B_1$ is decomposable ( $\leftrightarrow (1,0) \otimes v_2 \otimes v_3$).

Do the same thing on the bottom face and get $B_2$ decomposable. If we then look at

$$B - B_1 - B_2 = \left\{ \begin{matrix} \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \end{matrix} \right\}$$

(both $*$'s are $\neq 0$ , top one is $\frac{\det M_1}{x}$, bottom one is $\frac{\det M_2}{y}$.) It is easy to see that this last matrix is decomposable (add multiple of top row to bottom to get rid of the $*$).

**Final Case:** In this case

$$B = \left\{ \begin{matrix} M_1 \\ \\ M_2 \end{matrix} \right\}$$

where

$$M_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}$$

where $ab \neq 0$ and $cd \neq 0$. Now add bottom to top, that doesn't change the tensor rank and we are no longer in this situation. Done.

It is possible to show that the closure of the tensors of tensor rank $= 2$ is everything! (one does the elimination). But the following box

$$B = \left\{ \begin{matrix} M_1 \\ M_2 \end{matrix} \right\}$$

where we have

$$M_1 = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$$

with $abcd \neq 0$, has tensor rank exactly 3. This is very different from the case of rank for matrices. There are not equations to describe "tensor rank $\leq 2$", for if there were, that set would be closed and you wouldn't be able to "escape from it" to tensors of higher tensor rank.

## Geometric Formulation

Clearly, these questions about "tensor rank" can all be formulated in terms of questions about the nature of the Secant variety of the various Segre embeddings

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \longrightarrow \mathbb{P}(V_1 \otimes \cdots \otimes V_t) \simeq \mathbb{P}^N$$

where $N = -1 + \Pi_{i=1}^{t}(n_i + 1)$. Again the questions are: do the various Secant varieties have the "expected" dimension?

We have already seen that there are some "easy" cases where this does not happen, namely in the case of two vector spaces. (So, in our paradigm of the Veronese varieties, this is the analogue of the quadratic Veronese varieties, which all have deficient secant varieties.)

In a paper with Catalisano and Gimigliano we showed that "unbalanced products" are always defective. More precisely: let

$$\mathbb{X} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \times \mathbb{P}^{n}$$

then for $s$ an integer such that

$$\Pi_{i=1}^{t}(n_i+1)-(\sum_{i=1}^{t} n_i)+1 \leq s \leq \min\{n, -1+\Pi_{i=1}^{t}(n_i+1)\}$$

then $Sec_{s-1}(\mathbb{X})$ is defective.

**Example:** Consider $Sec_3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^{26}$. This should fill the space but it does not.

Also, $\mathbb{X} = \mathbb{P}^2 \times \mathbb{P}^n \times \mathbb{P}^n$ for $n$ even. Strassen showed that $U_E$ fills $\mathbb{P}^N$ for the first time when $E = (3/2)n + 2$ . This is later than it ought to fill.

E.g. for $n = 4$ we have $\mathbb{P}^2 \times \mathbb{P}^4 \times \mathbb{P}^4 \subset \mathbb{P}^{74}$. For $s = 8$ we should have $Sec_7(\mathbb{X}) = \mathbb{P}^{74}$ but it is not.

On the positive side, for $\mathbb{X} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ and $t \geq 3$, if

$$\left\lceil \frac{(n_1+\cdots+n_t+1)}{2} \right\rceil \geq \max\{n_t+1, s\}$$

then $Sec_{s-1}(\mathbb{X})$ is not defective.

How does one attack these problems? Without entering into any details, one is reduced to considering the multigraded Hilbert function of ideals of the form

$$\wp_1^2 \cap \cdots \cap \wp_s^2$$

where $\wp$ is the multihomogeneous prime ideal associated to a point in the product space.

Although we could handle that in certain cases, these multihomogeneous ideals were a bit difficult to deal with. So, in one of our papers we proposed to move from the multihomogeneous setting to a simple homogeneous setting. The gain in being able to deal with homogeneous ideals in a single polynomial ring was compensated by the fact that we now had to deal with more complicated ideals!

E.g. consider the problem of deciding the dimension of

$$Sec_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1).$$

We need to consider the Hilbert function of the ideal

$$(\wp_1^2 \cap \wp_2^2 \cap \wp_3^2 \cap q_1^3 \cap \cdots \cap q_4^3)_4 \subset k[x_0, x_1, ..., x_4]_4$$

In this case we have that the secant variety is defective if and only if this ideal contains two or more independent quartics.

Recently, with Catalisano and Gimigliano, we found the first infinite family in which all secant varieties are now known.

**Theorem:** Let $\mathbb{X}_t = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, $t \geq 3$ times.

Then, apart from $Sec_2(\mathbb{X}_4)$ all the other secant varieties for $\mathbb{X}_t$ (for every $t \geq 3$) is not defective.

**Aside:** These result has spawned a whole series of questions which have captured the attention of numerous mathematicians recently.

E.g.

1) What is the Hilbert function of $R/I$ when

$$I = \wp_1^{n_1} \cap \cdots \cap \wp_t^{n_t}.$$

and the $\wp_i \leftrightarrow P_i$ are general points of $\mathbb{P}^n$ (even for $n = 2$).

2) What are the graded Betti numbers for such ideals, when the points $P_i$ are general.

3) It is known what all the possibilities are for the Hilbert functions of points in $\mathbb{P}^n$, but it not known what are all the possibilities for "2-fat" points, even in $\mathbb{P}^2$. Nor even upper or lower bounds.

4) What are the equations for the Secant Varieties of the Veronese Varieties, for the varieties of reducible forms, for the Segre varieties. There has been some

work of Weyman, Landsberg and Manivel on the equations for small secant varieties with 3 or 4 factors and by Catalisano, Geramita and Gimigliano for unbalanced products of Segre varieties.

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