Summary of Last Lecture

Terracini's Lemma

Let $\mathbb{X} = J(\mathbb{X}_0, \ldots, \mathbb{X}_s)$, $\mathbb{X}_i \subset \mathbb{P}^n$. Let $P_i \in \mathbb{X}_i$ be general points and let

$$\mathbb{P}^s = \langle P_0, \ldots, P_s \rangle$$

Then, for a general point $Q \in \mathbb{P}^s$ we have

$$T_{Q,\mathbb{X}} = \langle T_{P_0,\mathbb{X}_0}, \ldots, T_{P_s,\mathbb{X}_s} \rangle$$

Example 1: Let $R = k[x_0, \ldots, x_3]$ and let

$$\mathbb{X}_0 = V_{(3,1),3}, \quad \mathbb{X}_1 = V_{(2,1,1),3}$$

i.e. $\lambda_1 = (3,1)$ and $\lambda_2 = (2,1,1)$ are partitions of 4.

So,

$$V_{(3,1),3} = \{[F] \mid F \in R_4, F = F_1F_2, \deg F_1 = 3, \deg F_2 = 1\}$$

$$V_{(2,1,1),3} = \{[F] \mid F \in R_4, F = F_1F_2F_3, \deg F_1 = 2, \deg F_2 = 1, \deg F_3 = 1\}$$

So, let $P_0 = [F_1F_2] \in V_{(3,1),3} = \mathbb{X}_0, I = (F_1, F_2)$ then

$$T_{P_0,\mathbb{X}_0} = \mathbb{P}(I_4).$$

If $P_1 = [QL_1L_2] \in V_{(2,1,1),3} = \mathbb{X}_1,\ I' = (L_1L_2, QL_2, QL_1)$, then

$$T_{P_1,\mathbb{X}_1} = \mathbb{P}(I'_4).$$

Thus, if we let $I'' = (F_1, F_2, L_1L_2, QL_2, QL_1)$ then

$$\dim J(X_0, X_1) = \mathbb{P}(I''_4)$$

where F_1, F_2, L_1, L_2, Q are general forms of the appropriate degrees.

Example 2: Let $\mathbb{X} = \nu_3(\mathbb{P}^2) \subset \mathbb{P}^9$. In this example, let $R = k[x_0, x_1, x_2]$. We have

$$\nu_3 : \mathbb{P}(R_1) = \mathbb{P}^2 \longrightarrow \mathbb{P}(R_3) = \mathbb{P}^9.$$

I want to find $\dim Sec_2(\nu_3(\mathbb{P}^2))$. Expected dimension is 8.

By Terracini, let P_0, P_1, P_2 be three points of $\mathbb{X}$, so

$$P_0 = [L_0^3], \quad P_1 = [L_1^3], \quad P_2 = [L_2^3].$$

$$T_{P_0,X} = \{[F] \in \mathbb{P}(R_3) \mid F = L_0^2 M, \quad M \in R_1\}$$

$$T_{P_1,X} = \{[F] \in \mathbb{P}(R_3) \mid F = L_1^2 M, \quad M \in R_1\}$$

$$T_{P_2,X} = \{[F] \in \mathbb{P}(R_3) \mid F = L_2^2 M, \quad M \in R_1\}$$

If we let $I = (L_0^2, L_1^2, L_2^2)$ then Terracini's Lemma says that

$$\dim(Sec_2(\nu_3(\mathbb{P}^2))) = \dim I_3 - 1$$

where L_0, L_1, L_2 are general linear forms. But, notice that, wlog, we can choose

$$L_0 = x_0, \quad L_1 = x_1, \quad L_2 = x_2$$

so that $I = (x_0^2,\ x_1^2,\ x_2^2)$.

It follows that $(R/I)_3 =< \overline{x_0 x_1 x_2} >$. So, $\dim I_3 = 9$ and hence the dimension of $Sec_2(\nu_3(\mathbb{P}^2)) = 8$, as was expected.

Inverse Systems

Consider, for the moment, two polynomial rings

$$R = k[x_0, \ldots, x_n] \qquad \text{and} \qquad S = k[y_0, \ldots, y_n].$$

I will think of R as a ring and S as a module over R by thinking of the elements of R as differential operators which act on the elements of S. More explicitly

$$x_i \circ y_j = (\partial/\partial y_i)(y_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

(so the x_i and y_j act like dual bases).

We can extend this action linearly to

$$R_i \times S_j \longrightarrow S_{j-i}.$$

E.g.

$$(x_1^2 + x_1x_2) \circ (y_0^3 + y_1^3) = x_1^2 \circ (y_0^3 + y_1^3) + x_1x_2 \circ (y_0^3 + y_1^3)$$

$$= (\partial^2/\partial y_1 \partial y_1)(y_0^3 + y_1^3) + (\partial^2/\partial y_1 \partial y_2)(y_0^3 + y_1^3) = 6y_1 + 0 = 6y_1.$$

Notice that the action of R on S **lowers** degrees and so S is not a finitely generated R-module.

If we write x^α, ($\alpha = (a_0, \ldots, a_n)$) to represent a monomial of R, and y^β, $\beta = (b_0, \ldots, b_n)$, a monomial of S then we say that

$$\alpha \leq \beta \Leftrightarrow a_i \leq b_i \ \forall i \Leftrightarrow x^\alpha \mid x^\beta.$$

The following Lemma is then clear.

Lemma:

$$x^\alpha \circ y^\beta = \begin{cases} 0 & \text{if } \alpha \text{ is not} \leq \text{to } \beta \\ \Pi_{i=0}^n (\left(\frac{(b_i)!}{(b_i - a_i)!}\right) y^{\beta - \alpha} & \text{if } \alpha \leq \beta \end{cases}$$

From this lemma we see that

$$R_j \times S_j \longrightarrow k$$

is a perfect pairing, i.e. the induced maps

$$\phi_1 : R_j \longrightarrow Hom_k(S_j, k) = S_j^* \text{ given by } r_j \longrightarrow r_j \circ -$$

and

$$\phi_2 : S_j \longrightarrow Hom_k(R_j, k) = R_j^* \text{ given by } s_j \longrightarrow - \circ s_j$$

are isomorphisms.

With these notions in hand, we define

Definitions: Let $V \subset R_j$. Then

$$V^\perp = \{s \in S_j \mid \phi_2(s)(V) = 0\}$$

2) Similarly if $W \subset S_j$, then $W^\perp \subset R_j$,

$$W^\perp = \{r \in R_j \mid \phi_1(r)(W) = 0\}$$

The following proposition is a standard fact about perfect pairings.

Proposition: Let $R_j \times S_j \longrightarrow k$ be as defined above. If $V \subset R_j$, $\dim V = t$, then

$$\dim V^{\perp} = \dim S_j - t.$$

We are now ready to define inverse systems.

Definition: Let I be an ideal of R. The *inverse system of* I, denoted I^{-1}, is the R-submodule of S consisting of all the elements of S that are annihilated by I.

Remarks:

1) Let $I = (F_1, \ldots, F_s)$ and let $G \in S$. By definition $G \in I^{-1}$ if and only if $F_i \circ G = 0$ for $i = 1, \ldots, s$. So, finding I^{-1} is like finding all the polynomial solutions to a system of partial differential equations.

2) Notice that if I is a homogeneous ideal of R then I^{-1} is a graded R-submodule of S.

Example: Let $I = (x_1) \subset k[x_1, x_2]$. Then

$$I^{-1} = \{G \in S \ | \ (\partial/\partial y_1) \circ G = 0\}$$

Now I is homogeneous so, let's look at I^{-1} in various degrees.

deg 1: $ay_1 + by_2 \in S_1$. Then $(\partial/\partial y_1)(ay_1 + by_2) = a$. So, $(I^{-1})_1 = (y_2)_1$.

deg 2: $ay_1^2 + by_1y_2 + cy_2^2$ is annihilated by $(\partial/\partial y_1)$ if and only if $a = b = 0$. So, $(I^{-1})_2 = (y_2^2)_2$. Continuing in this way we find

$$I^{-1} = k \oplus < y_2 > \oplus < y_2^2 > \oplus < y_2^3 > \oplus \cdots$$

Note: I^{-1} is not a finitely generated submodule of S.

How do we find I^{-1} in general?

From the pairing $R \times S \longrightarrow S$, we get

$$\begin{array}{ccc} R_j \times S_j & \longrightarrow & k \\ \cup & & \\ I_j \times I_j^{\perp} & \longrightarrow & 0 \end{array}$$

i.e. I_j certainly annihilates exactly $I_j^{\perp}$, but maybe other things in I don't annihilate stuff in $I_j^{\perp}$? So, we can certainly say that

$$(I^{-1})_j \subset I_j^{\perp}$$

The interesting thing is that we have equality here. I.e.

Proposition:

$$(I^{-1})_j = (I_j)^{\perp}$$

Proof: It will be enough to show that I annihilates everything in $I_j^{\perp}$.

Consider I_t for $t > j$. Then $I_t \times I_j^{\perp} \longrightarrow S_{j-t}$ which is 0, since $j - t < 0$.

Now consider $F \in I$, where $\deg F = d < j$. Let x^α be any monomial of degree $j-d$. Then $x^\alpha F \in I_j$ and so $(x^\alpha F) \circ G = 0$ for all $G \in I_j^\perp$.

But $(x^\alpha F) \circ G = x^\alpha \circ (F \circ G) = 0$. But, $F \circ G \in S_{j-d}$ and we just saw that it annihilates every monomial x^α in R_{j-d}. But, we have a perfect pairing and so this means that $F \circ G = 0$, as we wanted to show.

We get an immediate and useful corollary from this.

Corollary:

$$\dim(I^{-1})_j = \dim(R_j/I_j) = H_{R/I}(j).$$

The usefulness of this observation comes from the fact that we can occasionally use this to find the Hilbert function of an ideal by finding the Hilbert function of its inverse system in that degree (or vice versa).

Observations:

1) I^{-1} is a finitely generated R-module if and only if R/I is an artinian ring, i.e. the dimension of R/I as a k-vector space, is finite.

This is clear since in order for an R-submodule of S to be finitely generated, the module must be zero in all large degrees. This forces its perp to be everything in R from some degree on, i.e. $I_d = R_d$ for all $d >> 0$.

2) It is useful to imagine what I^{-1} is for a monomial ideal.

Inasmuch as the monomials of degree d in R and the monomials of degree d in S are practically a dual basis (there are the coefficients) we see that I_d^{-1} consists of all the monomials y^β for which x^β is not in I.

The following is easily proved

Proposition: If I and J are ideals of R then

$$(I \cap J)^{-1} = I^{-1} + J^{-1}$$

This follows easily from the vector space equality

$$(U_1 \cap U_2)^\perp = U_1^\perp + U_2^\perp$$

and the fact that the inverse system of an ideal is formed component-wise.

I now want to make a non-trivial calculation of an inverse system which is connected to our problem of finding the dimensions of the Secant Varieties to the Veronese varieties.

Example: Let $R = \mathbb{C}[x_0, \ldots, x_n]$ and let $\wp = (L_1, \ldots, L_n)$. Then $\wp \leftrightarrow P \in \mathbb{P}^n$.

After an invertible change of coordinates in R and $\mathbb{P}^n$ we can take $\wp = (x_1, \ldots, x_n)$ and $P = [1 : 0 : \ldots : 0]$.

Let $I = \wp^{\ell+1}$. Then I is a monomial ideal and I^{-1} is the R-submodule of S generated by $\{y^\beta \mid x^\beta \notin I\}$.

Let's write down I^{-1} in detail.

1) Since $I_t = 0$ for $t \leq \ell$ we get that $I_t^{-1} = S_t$ for $t \leq \ell$.

For convenience in the description of the rest of I^{-1} let's denote by $T = k[y_1, \ldots, y_n]$.

Let $t \geq \ell + 1$ then, by keeping track of the monomials by noting the power of y_0 that a monomial can contain, we can write S_t as:

$$S_t =< y_0^t > \oplus < y_0^{t-1} T_1 > \oplus \cdots \oplus < y_0^{t-\ell} T_\ell > \oplus$$

$$\left[< y_0^{t-(\ell+1)} T_{\ell+1} > \oplus \cdots \oplus T_t\right]$$

Notice that the part of this expression on the second line corresponds to the monomials which are in $(\wp^{\ell+1})_t$ and hence

$$[(\wp^{\ell+1}]_t^{-1} =< y_0^t > \oplus < y_0^{t-1} T_1 > \oplus \cdots \oplus < y_0^{t-\ell} T_\ell >$$

$$= y_0^{t-\ell} S_\ell$$

We thus obtain the following result:

Proposition: Let $\wp = (x_1, \ldots, x_n) \subset \mathbb{C}[x_0, \ldots, x_n]$ and let $S = \mathbb{C}[y_0, \ldots, y_n]$. If $\ell \geq 0$ then

$$(\wp^{\ell+1})^{-1} = S_0 \oplus \cdots \oplus S_\ell \oplus y_0 S_\ell \oplus y_0^2 S_\ell \oplus \cdots$$

More generally, if $P = [p_0 : \ldots : p_n] \in \mathbb{P}^n$ and $P \leftrightarrow \wp$, let

$$L_P = p_0 y_0 + \cdots + p_n y_n \in S.$$

Then

$$(\wp^{\ell+1})^{-1} = S_0 \oplus S_1 \oplus \cdots \oplus S_\ell \oplus L_P S_\ell \oplus L_P^2 S_\ell \oplus \cdots$$

Now, let's put this all together. We have

$$\nu_d : [L] \longrightarrow [L^d], \ \ \mathbb{P}(R_1) \longrightarrow \mathbb{P}(R_d)$$

the Veronese map.

The tangent space at the point $[L^d]$ is the projectivization of $R_1 L^{d-1} \subset R_d$. Thus, by Terracini's Lemma: if we choose $[L_0^d], [L_1^d], \cdots, [L_t^d]$ as general points of $\nu_d(\mathbb{P}(R_1))$, then

$$\dim Sec_t(\nu_d(\mathbb{P}^n)) = \dim_{\mathbb{C}} < L_0^{d-1}R_1, \ldots, L_t^{d-1}R_1 > -1.$$

But,

$$\dim_{\mathbb{C}} < L_0^{d-1}R_1, \ldots, L_t^{d-1}R_1 >=$$

$$= \dim_{\mathbb{C}} < \overline{L_0}^{d-1}S_1, \ldots, \overline{L_t}^{d-1}S_1 >$$

where if $L = a_{i0}x_0 + \cdots + a_{in}x_n$ then $\overline{L} = a_{i0}y_0 + \cdots + a_{in}y_n$.

Also, $\dim_{\mathbb{C}} \overline{L_i}^{d-1}S_1 = \dim_{\mathbb{C}}(\wp_i^2)^{-1}$, where $\wp_i \leftrightarrow P_i$, where

$$P_i = [a_{i0} : \ldots : a_{in}] \ .$$

It follows that

$$\dim_{\mathbb{C}} < \overline{L_0}^{d-1} S_1, \ldots, \overline{L_t}^{d-1} S_1 >= \dim_{\mathbb{C}} (\wp_0^2 \cap \cdots \cap \wp_t^2)_d^{-1} =$$

$$H\left(\frac{R}{\wp_0^2 \cap \cdots \cap \wp_t^2}, d\right).$$

Theorem: Let $P_0, \ldots, P_t$ be general points in $\mathbb{P}^n$ and suppose that $P_i \leftrightarrow \wp_i \subset R = \mathbb{C}[x_0, \ldots, x_n]$. Then,

$$\dim Sec_t(\nu_d(\mathbb{P}^n)) = H\left(\frac{R}{\wp_0^2 \cap \cdots \cap \wp_t^2}, d\right) - 1$$

So, this whole procedure brings us to the study of intersections of ideals of the form $\wp^2$, where $\wp$ is the ideal of a point. How can we think of such ideals?