The Veronese Varieties

These varieties are based on the structure of the polynomial ring $Sym(V^*) = R = \mathbb{C}[x_0, \ldots, x_n] = \oplus_{d \geq 0} R_d$.

There is a very simple function

$$\begin{array}{ccc} \mathbb{P}^n = \mathbb{P}(R_1) & \longrightarrow & \mathbb{P}(R_d) = \mathbb{P}^N, \quad N = \binom{d+n}{n} - 1, \\ [L] & \longrightarrow & [L^d] \end{array}$$

Definition: The image of this map is a projective variety of dimension n called the d^{th}*-Veronese embedding of* $\mathbb{P}^n$.

(Hartshorne calls this the *d-uple embedding of* $\mathbb{P}^n$).

Example: Let's look at the case: $n = 1$, $d = 3$. We have $R = \mathbb{C}[x_0, x_1]$ and the map is

$$[a_0x_0 + a_1x_2] \longrightarrow [(a_0x_0 + a_1x_2)^3]$$

$$= [a_0^3x_0^3 + 3a_0^2a_1x_0^2x_1 + 3a_0a_1^2x_0x_1^2 + a_1^3x_1^3]$$

If we use as a basis for R_d the monomials of degree d and order them lexicographically and write things with respect to coordinates we get that the map is defined by:

$$[a_0 : a_1] \longrightarrow [a_0^3 : 3a_0^2 a_1 : 3a_0 a_1^2 : a_1^3]$$

Notice that by performing a collineation on $\mathbb{P}^3$ we can change the image to

$$[a_0^3 : a_0^2 a_1 : a_0 a_1^2 : a_1^3]$$

Oftentimes this is the may the Veronese embeddings are defined, namely:

take all of the monomials of degree d in R and order them lexicographically. Call them

$$M_0, M_1, \dots, M_N$$

with N as above. The embedding can be described by

$$P = [a_0 : \dots : a_n] \longrightarrow [M_0(P) : M_1(P) : \dots : M_N(P)]$$

It is clear that the two methods of describing the Veronese embeddings are equivalent in characteristic zero.

The Quadratic Veronese Embeddings

These are a very nice special case which I would like to study first. We have $R = \mathbb{C}[x_0, \ldots, x_n]$, and

$$\begin{array}{ccc} \mathbb{P}^n = \mathbb{P}(R_1) & \longrightarrow & \mathbb{P}(R_2) = \mathbb{P}^s, \quad s = \binom{d+2}{2} - 1 \\ [L] & \longrightarrow & [L^2] \end{array}$$

We denote the image by $V_{2,n}$. So, the Veronese variety consists of forms of degree 2 which are the square of a linear form. What do the various secant varieties of it look like?

Clearly, $[F]$ belongs to the secant line which connects the point $P_0 = [L_0^2]$ to the point $P_1 = [L_1^2]$ if and only if $F = [\alpha_0 L_0^2 + \alpha_1 L_1^2]$. More generally, $[F]$

belongs to a secant $\mathbb{P}^r$ to $V_{2,n}$ if and only if

$$F = \beta_0 N_0^2 + \ldots + \beta_r N_r^2, \text{ where } \beta_i \in \mathbb{C}, \quad N_i \in R_1.$$

But, it is well known that the quadratic polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ are in 1-1 correspondence with the symmetric $(n+1) \times (n+1)$ matrices and that symmetric matrices are diagonalizable. What this means in our context is that there exists a basis, $L_0, \ldots, L_n$ for R_1 so that with respect to that basis the quadratic form we began with can be written

$$F = L_0^2 + \ldots + L_t^2$$

where $t+1$ is the rank of the associated symmetric matrix.

Thus, if we think of $\mathbb{P}^s$ ($s = \binom{d+2}{2} - 1$) as the projectivization of the vector space of all $(n+1) \times (n+1)$ symmetric matrices, then

$$Sec_t(V_{2,n}) = \{ \text{ symmetric matrices of rank } \leq t+1\}.$$

I.e. these are the symmetric matrices of size $(n+1) \times (n+1)$ for which all the $t+2$-minors vanish.

Let's look at the very special case of $n = 2$. In this case $R = \mathbb{C}[x_0, x_1, x_2]$ and the quadratic Veronese embedding of $\mathbb{P}^2$ is

$$\phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^5$$

where we think of $\mathbb{P}^5$ as the 3×3 symmetric matrices, up to multiplication by a non-zero scalar.

We have that $V_{2,2} \subset \mathbb{P}^5$ is a surface, isomorphic to $\mathbb{P}^2$. It can be described by the ideal of all the 2×2 minors of the generic symmetric matrix. $Sec_1(V_{2,2})$ can thus be thought of as all the 3×3 symmetric matrices of rank 2.

But, saying that a 3×3 matrix has rank 2 is equivalent to saying that its determinant is 0. Thus $Sec_1(V_{2,2})$ can be described as the zeroes of the determinant of the generic 3×3 symmetric matrix. I.e. it is a cubic hypersurface in $\mathbb{P}^5$ and, as such, it has dimension 4. But, the expected dimension of $Sec_1(V_{2,2})$ is $\min\{2 \cdot 2 + 1, 5\} = 5$. So, this classical variety is defective as well.

There are many interesting things going on here. In the first place, it is not hard to show that **ALL** the quadratic Veronese varieties whose secant variety is a proper subvariety of the envelopping $\mathbb{P}^s$ are defective.

Interestingly enough, it is a well known theorem of Severi, that the only smooth surface in $\mathbb{P}^5$ whose secant line variety is not all of $\mathbb{P}^5$ (i.e. is defective) is $V_{2,2}$.

I want to describe two further classes of projective varieties whose secant varieties are interesting and which we would like to understand better. The reasons for this are many but involve a combination of the fact that the results would be interesting to understand on their own and also the solutions to these problems have interesting applications not only in mathematics but in other areas as well.

The Varieties of Reducible Forms

These varieties are defined as follows: fix $n \geq 1$ and consider the polynomial ring

$$R = \mathbb{C}[x_0, \ldots, x_n] = \oplus_{j \geq 0} R_j .$$

Fix any positive integer $d > 1$. A *partition* λ *of* d is a tuple of positive integers, $\lambda = (d_1, d_2, \ldots, d_s)$, where

$$i) \quad d_1 \geq d_2 \geq \cdots \geq d_s \ ;$$

$$ii) \quad \textstyle\sum_{i=1}^{s} d_i = d \ .$$

If λ is a partition of d we write $\lambda \vdash d$.

Given $n,\ d,\ \lambda$ (as above) we define

$$V_{n,\lambda} = \{[F] \in \mathbb{P}(R_d) \ \mid \ F = F_1 \cdots F_s, \deg F_i = d_i \ \}.$$

It's easy to see that

$$\dim V_{n,\lambda} = (\sum_{i=1}^{s} \binom{d_i + n}{n}) - s.$$

We will see later that the joins and secant varieties of these varieties have interesting applications to an extension of a theorem of Noether and Severi about the existence of complete intersection subvarieties on general hypersurfaces of degree d in $\mathbb{P}^n$.

The Segre-Veronese Varieties

As the name suggests, these varieties are a mix of Segre and Veronese varieties. Very little is known about them even though a knowledge of the dimensions of the secant varieties to them has interesting applications in Algebraic Geometry as well as applications outside the area. Let me quickly give a description of these varieties. I hope we will have time to mention some of the open problems concerning them. (Perhaps in the Tutorial?)

Let

$$\mathbb{X} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s} = \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_s)$$

where $\dim V_i = n_i + 1$. Let $(d_1, \ldots, d_s)$ be an s-tuple of positive integers.

We define a map

$$\phi : \mathbb{X} \xrightarrow{(d_1,\ldots,d_s)} \mathbb{P}(Sym_{d_1}(V_1) \otimes \cdots \otimes Sym_{d_s}(V_s))$$

by taking the composition of the appropriate Veronese maps into the product and then following it by the Segre map to the tensor product.

E.g. Consider $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{(2,2)} \mathbb{P}(Sym_2(V_1) \otimes Sym_2(V_2))$. We will suppose that $V_1 = \langle x_0, x_1 \rangle$ and $V_2 = \langle y_0, y_1 \rangle$. The the map $(2,2)$ is given by:

$$[x_0 : x_1] \times [y_0, y_1] \longrightarrow [x_0^2 : x_0x_1 : x_1^2] \times [y_0^2 : y_0y_1 : y_1^2] \longrightarrow$$

$$[x_0^2 \otimes y_0^2 : x_0^2 \otimes y_0y_1 : x_0^2 \otimes y_1^2 : \ldots : x_1^2 \otimes y_0^2 : x_1^2 \otimes y_0y_1 : x_1^2 \otimes y_1^2]$$

(which is in $\mathbb{P}^8$).

We can look at this (as we did earlier) as giving us a 3×3 matrix, whose entries are all the monomials in the two sets of variables x_0, x_1, y_0, y_1 of bi-degree $(2,2)$.

The Search for the dimensions of Joins and Secant Varieties

Up to this stage we have used rather ad-hoc methods for trying to figure out the dimensions of the higher order secant varieties to some simple varieties. I would like to now explain a method that we can use very generally, that is the major tool which gets us started in trying to find the dimensions of Secant Varieties. This is the **Lemma of Terracini**. (Alessandro Terracini (1889-1968) was an Italian mathematician who shortly after obtaining a cattedra at the University of Torino was forced to leave Italy in 1938. He went to live in Argentina but returned to Italy in 1948. He was, by this time, greatly appreciated by the Italian mathematical community which reveled in his return.)

Terracini's Lemma is really an observation that he made about the tangent space to a general point on a join. Of course, if our varieties are not too bad, the general point on the join will be a smooth point, and assuming again that our varieties are connected, the joins will be connected as well. Consequently, knowing the dimension of the tangent space at a general point of the join is the same thing as knowing the dimension of the join.

So, let $\mathbb{X}_i,\ i = 0, \ldots, s$ $(s < n)$ be a family of non-degenerate varieties in $\mathbb{P}^n$ of dimensions d_i. If $P_i \in \mathbb{X}_i$ is a general point of $\mathbb{X}_i$ and $T_{P_i,\mathbb{X}_i}$ is the *projectivized* tangent space to $\mathbb{X}_i$ at P_i, then:

if $P \in \mathbb{P}^s =< P_0, \ldots, P_s >\subset J(\mathbb{X}_0, \ldots, \mathbb{X}_s) = \mathbb{Y}$, is a general point of $\mathbb{Y}$ we have that

$$T_{P,\mathbb{Y}} = \langle T_{P_0,\mathbb{X}_0}, \ldots, T_{P_s,\mathbb{X}_s} \rangle$$

i.e. the projective space spanned by these tangent spaces.

This is Terracini's Lemma; it provides us with a method to find the dimensions of the various joins we have defined, assuming of course that we are able to calculate the tangent spaces at general points of our varieties and then figure out the dimension of the linear space they span.

It is an easy exercise (which I suggest you try) to see that $\dim J(\mathbb{X}_0, \ldots, \mathbb{X}_s)$ is the expected dimension (i.e. is $s + \sum_{i=0}^{s} \dim \mathbb{X}_i$ if this is $\leq n$) precisely when these different tangent spaces don't intersect at all. So, the expected dimension depends on a "general position" argument for tangent spaces at points of our varieties.

Thus, in order to apply Terracini's Lemma we will have to figure out a way to find these various tangent spaces.

Let's begin with one of the last families of varieties I introduced above, the varieties of reducible forms. In some sense their tangent spaces are the easiest to understand and it will give us a clue as to how to confront the same kind of problem for the other varieties.

Let $V_{\lambda,n} \subset \mathbb{P}(R_d)$, $R = \mathbb{C}[x_0, \ldots, x_n]$, $\lambda = (d_1, \ldots, d_s)$, $\sum_{i=1}^{s} d_i = d$. Let P be a point in $V_{\lambda,n}$. Then, without loss of generality we can suppose that

$$P = [F],\ F = F_1 F_2 \cdots F_s, \quad \deg F_i = d_i, \quad F_i \text{ all irreducible.}$$

How do we find $T_{P,V_{\lambda,n}}$?

It will be easier to calculate the tangent space to the point F on the affine cone over $V_{\lambda,n}$, i.e. in the affine space R_d.

Recall that we have a map

$$\begin{array}{ccccc} \mathbb{C}^\ell & = & R_{d_1} \times \cdots \times R_{d_s} & \longrightarrow & R_d, \qquad \ell = \sum_{i=1}^{s} \binom{d_i+n}{n} \\ & & (G_1, \cdots, G_s) & \longrightarrow & G_1 \cdots G_s \end{array}$$

We are interested in the tangent space at the image of the point $(F_1, \ldots, F_s) \in \mathbb{C}^\ell$.

Recall that all the vectors in the tangent space at the point $F_1 \cdots F_s$ (in the image of this map) are the tangent vectors to the images of curves in $\mathbb{C}^\ell$ which pass through $(F_1, \ldots, F_s)$ in all the directions permitted by the tangent space to $\mathbb{C}^\ell$ at that point.

So, what we have to do is find curves through $(F_1, \ldots, F_s)$ which have all possible tangent vectors, take their images under the map above and calculate the tangent vectors to those images at the point $F_1 \cdots F_s$.

At any point of $\mathbb{C}^\ell$, the tangent space is precisely $\mathbb{C}^\ell$ and so we can represent every direction from $(F_1, F_2, \ldots, F_s)$ by a tuple $(F_1', F_s', \ldots, F_s')$ and the curve with that tangent direction is nothing more than the parametrized line

$$(F_1, F_2, \ldots, F_s) + t(F_1', F_2', \ldots, F_s')$$

The image of this curve is the parametrized curve

$$(F_1 + tF_1')(F_2 + tF_2') \cdots (F_s + tF_s').$$

and, by Taylor's Theorem, it's tangent vector at the point $F_1 F_2 \cdots F_s$ is the coefficient of t in the equation of the curve.

That coefficient is nothing other than

$$\sum_{i=1}^{s} F_1 \cdots F_{i-1} F_i' F_{i+1} \cdots F_s \ .$$

I.e. the affine tangent space to the cone over $V_{\lambda,n}$ at the point $F = F_1 F_2 \cdots F_s$ is the following subspace of R_d,

$$R_{d_1}(F_2 \cdots F_s) + R_{d_2}(F_1 F_3 \cdots F_s) + \cdots + R_{d_s}(F_1 F_2 \cdots F_{s-1}) \ ,$$

i.e. the degree d homogeneous piece of the ideal

$$I = (F_2 \cdots F_s, F_1 F_3 \cdots F_s, \ldots, F_1 F_2 \cdots F_{s-1}) \ .$$

Passing to the projectivized tangent space, we get that

$$\dim V_{\lambda,n} = \dim T_{P,V_{\lambda,n}} = \dim I_d - 1.$$

But, more important than knowing the dimension of $V_{\lambda,n}$ (which we already knew!) is the fact that we now have the tools available to use Terracini's Lemma.

Let's now do that for some special varieties of reducible forms. In particular, let $\lambda_0, \ldots, \lambda_s$ be partitions of d into 2 parts, i.e.

$$\lambda_0 = (d_{01}, d_{02}), \ldots, \lambda_s = (d_{s1}, d_{s2}) \ .$$

and let $\mathbb{X}_i = V_{\lambda_i,n}$.

I would like to find the dimension of

$$J = J(\mathbb{X}_0, \ldots, \mathbb{X}_s) \subset \mathbb{P}(R_d).$$

From what we just discovered above, it is enough to consider the ideal

$$I = (F_{01}, F_{02}, \ldots, F_{s1}, F_{s2})$$

where F_{i0}, F_{i1} are general forms having degrees, respectively, d_{i0}, d_{i1}, and figure out how big this ideal is in degree d.

It turns out that there is a very well known conjecture (due to Ralf Froberg) whose solution would answer this question. To explain Froberg's conjecture I want to remind you of another definition.

Let $R = \mathbb{C}[x_0, \ldots, x_n] = \oplus_{t\geq 0} R_t$, let I be a homogeneous ideal of R and set $A = R/I = \oplus_{t\geq 0} A_t = \oplus_{t\geq 0}(R_t/I_t)$.

Definition: The Hilbert function of A, $H_A : \mathbb{N} \longrightarrow \mathbb{N}$ is the function

$$H_A(r) = \dim_{\mathbb{C}}(A_r).$$

We know $\dim_{\mathbb{C}}(R_d) = \binom{d+n}{n}$, so knowing $H_A(d)$ is equivalent to knowing $\dim_{\mathbb{C}}(I_d)$.

Froberg's Conjecture says more than what we want to know. His conjecture is the following:

Let $H_1, \ldots, H_t$ be generic forms in R, $\deg H_i = d_i$. Then, for each $i \leq t$, the linear transformations

$$\overline{H_i} : R/(H_1, \ldots, H_{i-1})_s \longrightarrow R/(H_1, \ldots, H_{i-1})_{d_i+s}$$

is of maximal rank for every s and every i.

It is a simple matter (complicated arithmetic, however) to go from this statement to a statement about the Hilbert function of $A = R/(H_1, \ldots, H_t)$.

Since that was what we needed to know to find the dimension of the join J above, this is an interesting connection between an algebraic and a geometric problem. (Mention the work with Carlini and Chiantini)

Calculating the Secant Varieties of the Veronese Varieties

Recall that in the case of the Veronese varieties we had $R = \mathbb{C}[x_0, \dots, x_n] = \oplus_{d \geq 0} R_d$ and that the d^{th} Veronese variety was the image of the map

$$\begin{array}{ccccc} \mathbb{P}^n & = & \mathbb{P}(R_1) & \xrightarrow{\nu_d} & \mathbb{P}(R_d) = \mathbb{P}^N, \quad N = \binom{d+n}{n} - 1 \\ & & [L] & \longrightarrow & [L^d] \end{array}$$

Having acquired some expertise in the earlier case, let's pick $L \in \mathbb{C}^{n+1}$, since the tangent space to L in $\mathbb{C}^{n+1}$ is $\mathbb{C}^{n+1}$, we need to find lines through L in all these directions. Such a direction is given by an $M \in \mathbb{C}^{n+1}$ (which we can think of as a linear form). So, the lines we need to look at, through L, are

$$L + tM, t \in \mathbb{C}, M \in R_1.$$

Under ν_d, these lines are sent to $(L+tM)^d$, which we can expand as

$$L^d + t(L^{d-1}M) + t^2(L^{d-2}M^2) + ...$$

from which we see that the tangent space to L^d consists of all forms of the type ML^{d-1}, $M \in R_1$.

So, if we are interested in knowing e.g. $\dim Sec_t(\nu_d(\mathbb{P}^n))$ we need to know the dimension of the vector space

$$R_1L_0^{d-1} + R_1L_1^{d-1} + \cdots + R_1L_t^{d-1}$$

where $L_0, L_1, \ldots, L_t$ are a general set of $t+1$ linear forms. I.e. we need to know the size of the ideal

$$I = (L_0^{d-1}, \ldots, L_t^{d-1})$$

in degree d.

This turns out to not be such an easy problem to confront, and I would like to now explain another way to approach this problem. To do that I need to explain something about Macaulay's Inverse System.