Secant Varieties

Notation and Terminology:

Let V be an $n+1$ dimensional vector space over the field k (usually $\mathbb{C}$) then

$$\mathbb{P}(V) = \{ \text{ collection of all the 1-dimensional subspaces of } V \ \}$$

If $\mathcal{B} = \{e_0, \ldots, e_n\}$ is a basis for V and $0 \neq v = (a_0, \ldots, a_n)$ is the expression for v in coordinates with respect to the basis $\mathcal{B}$, then we write

$$\ell = \langle v \rangle = [a_0 : \ldots : a_n] = P \in \mathbb{P}(V),$$

to indicate the 1-dimensional subspace generated by v and refer to it as a *point* of $\mathbb{P}(V)$. Clearly these coordinates are only determined up to a non-zero constant. Once a basis is chosen we will often use the notation $\mathbb{P}^n$ to refer to $\mathbb{P}(V)$ and write the points of $\mathbb{P}^n$ as $[a_0 : \ldots : a_n]$.

Let V^* be the dual space to V and let $x_0, \ldots, x_n$ be a dual basis to $\mathcal{B}$. We write

$$Syn(V^*) = R = k[x_0, \ldots, x_n] = \oplus_{d \geq 0} R_d$$

where R_d is the k-subspace of R spanned by the monomials of degree d in $x_0, \ldots, x_n$. It has dimension $\binom{d+n}{n}$ as a k-vector space. An element $F \in R_d$ is a *homogeneous polynomial* of degree d (sometimes called a *form* of degree d).

Definition: The form F vanishes at the point

$$P = [a_0 : \ldots : a_n] \text{ if } F(a_0, \ldots, a_n) = 0.$$

> This definition is independent of the coordinates chosen for the point P, i.e. of the vector v which we choose to generate the 1-dimensional subspace ℓ.

If $f \in R$ is any element then we can write

$$f = F_0 + \cdots + F_r, \ F_i \text{ form of degree } i \text{ and } F_r \neq 0$$

uniquely. It is well known that f vanishes at P if and only if F_i vanishes at P for $0 \leq i \leq r$.

This leads us to consider *homogeneous ideals* in $\mathbb{R}$, i.e. ideals with the property that

$$f = F_0 + \cdots + F_r \in I \Leftrightarrow F_j \in I, \ 0 \leq j \leq r.$$

Given I homogeneous in R we define

$$Z(I) := \{P \in \mathbb{P}^n \ \mid f(P) = 0 \text{ for all } f \in I\ \}$$

We put a topology on $\mathbb{P}^n$ be declaring the closed sets to be the sets $Z(I)$ with I a homogeneous ideal in R. This is the *Zariski topology* on $\mathbb{P}^n$.

These closed sets are the *projective algebraic sets* (which I will sometimes sloppily refer to as *projective varieties*). (The more subtle object i.e. the projective scheme defined by the ideal I, I will avoid talking about for as long as possible since the scheme consists not only of a subset of $\mathbb{P}^n$, i.e. $Z(I)$, but also of a sheaf of rings on that subset. It is a more discriminating object.)

Now for some new definitions:

Definitions:

1) Let $P \neq Q$ be two points of $\mathbb{P}(V)$, $P = \langle v_1 \rangle$, $Q = \langle v_2 \rangle$. The *join of* P *and* Q, denoted $J(P,Q)$, is:

$$J(P,Q) := \{\langle \lambda v_1 + \delta v_2 \rangle \ \mid \ (\lambda, \delta) \neq (0,0)\}$$

Notice that this subset is really a copy of $\mathbb{P}^1$ inside $\mathbb{P}^n$ since the points of $J(P,Q)$ are in 1-1 correspondence with $[\lambda : \delta] \in \mathbb{P}^1$. This is the *line* which connects P to Q.

Now let $\mathbb{X}_0$ and $\mathbb{X}_1$ be two projective algebraic subsets of $\mathbb{P}^n$.

2) We define the *join of* $\mathbb{X}_0$ *and* $\mathbb{X}_1$ as follows:

$$J(\mathbb{X}_0, \mathbb{X}_1) := \overline{\cup\{J(P,Q) \ \mid \ P \neq Q, P \in \mathbb{X}_1, Q \in \mathbb{X}_2 \ \}}$$

where the "overline" indicates *closure* in the Zariski topology.

3) If $\mathbb{X}_1 = \mathbb{X}_2 = \mathbb{X}$ we write

$$J(\mathbb{X}_1, \mathbb{X}_2) = Sec_1(\mathbb{X})$$

and call this the *secant line variety associated to* $\mathbb{X}$.

More generally,

4) if $P_0 = \langle v_0 \rangle, \ldots, P_s = \langle v_s \rangle$ are $s+1$ points of $\mathbb{P}^n$ that are linearly independent (so $s \leq n$), then we define

$$J(P_0, \ldots, P_s) := \{ \ \ \langle \lambda_0 v_0 + \cdots + \lambda_s v_s \rangle \quad \mid \quad [\lambda_0 : \ldots : \lambda_s] \in \mathbb{P}^s \}$$

The join of these points is nothing other than the $\mathbb{P}^s \subset \mathbb{P}^n$ which is the linear span of the points $P_0, \ldots, P_s$.

Again, more generally, we have

5) If $\mathbb{X}_i$, $i = 0, \ldots, s$ are $s \leq n$ projective algebraic subsets of $\mathbb{P}^n$ then the join of $\mathbb{X}_0, \ldots, \mathbb{X}_s$ is

$$J(\mathbb{X}_0, \ldots, \mathbb{X}_s) =$$

$$\overline{\cup\{J(P_0, \ldots, P_s) \mid P_i \in \mathbb{X}_i, \text{ lin. ind. points}\}}$$

6) If $\mathbb{X}_0 = \mathbb{X}_1 = \cdots = \mathbb{X}_s = \mathbb{X}$ then we write

$$J(\mathbb{X}_0, \ldots, \mathbb{X}_s) = Sec_s(\mathbb{X})$$

and we call this the *variety of secant $\mathbb{P}^s$'s to* $\mathbb{X}$.

Examples:

1) In $\mathbb{P}^2$, if $\mathbb{X}_0 = P_0$ and $\mathbb{X}_1 = \{P_1, P_2\}$, then there are two possibilities for $J(\mathbb{X}_0, \mathbb{X}_1)$. It is either a union of two lines or one line, depending on whether or not the three points are collinear.

2) In $\mathbb{P}^2$, if $\mathbb{X}_0 = \{P_0\}$ and $\mathbb{X}_1$ is the line L, then there are two choices for $J(\mathbb{X}_0, \mathbb{X}_1)$. It is $\mathbb{P}^2$ if the point is not on the line, and it is the line L if the point is on the line.

As you can see from these examples, the following definition is useful.

Definition: We say that $\mathbb{X} \subset \mathbb{P}^n$ is *degenerate* if there does not exist at least one subset $\{P_0, \ldots, P_n\} \subset \mathbb{X}$ whose join is $\mathbb{P}^n$. Otherwise we say that $\mathbb{X}$ is *non-degenerate*.

Let's look at a more interesting example. We will see, with this example, why I added "closure" to the definition of the join.

Example: The rational normal curve in $\mathbb{P}^3$. This is the image, $\mathcal{C}$, of the map

$$\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

given by

$$[a : b] \longrightarrow [a^3 : a^2 b : ab^2 : b^3]$$

It is easy to check that this map is 1-1.

Claim: The image of this map is a curve of degree 3 in $\mathbb{P}^3$.

> Recall that the *degree of a curve in* $\mathbb{P}^n$ is the number of points we get when we intersect the curve with a general hyperplane of $\mathbb{P}^n$.

Let's use coordinates $Y_0, \ldots, Y_3$ for $\mathbb{P}^3$. Then a general hyperplane is given by

$$\alpha_0 Y_0 + \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3 = 0$$

with the $\alpha_i \in k$. Now, this hyperplane meets the curve $\mathcal{C}$ precisely when

$$F(a,b) = \alpha_0(a^3) + \alpha_1(a^2b) + \alpha_2(ab^2) + \alpha_3(b^3) = 0.$$

But, we can look at this as a homogeneous polynomial of degree 3 in the variables a and b. In $\mathbb{C}[a,b]$ a homogeneous polynomial of degree 3 factors completely into linear factors, i.e.

$$F(a,b) = (u_{1,1}a + u_{1,2}b)(u_{2,1}a + u_{2,2}b)(u_{3,1}a + u_{3,2}b)$$

which has exactly the three roots

$$[-u_{1,2}, u_{1,1}], [-u_{2,2} : u_{2,1}], [-u_{3,2} : u_{3,1}].$$

Now, we want to know which points of $\mathbb{P}^3$ lie on secant lines to $\mathcal{C}$ and which do not.

So, pick a point in $\mathbb{P}^3$, call it P. If it lies on $\mathcal{C}$ it certainly lies on a secant line, so let's suppose it is not on $\mathcal{C}$ itself. Also pick any plane, $\Pi \subset \mathbb{P}^3$ which doesn't contain the point P (draw the picture).

If we project the curve $\mathcal{C}$ from the point P, onto the plane Π, how can we tell if the point P was on a secant line to $\mathcal{C}$?

There will be two points on $\mathbb{C}$ which end up at the same place on the projected curve. I.e. it will show up as a nodal *singular point* on the projected curve.

But, what does the projected curve look like? It is a plane curve and it is clear that it also has degree 3, so it is a cubic plane curve. Moreover, it is the image of $\mathbb{P}^1$ so it is a *rational curve*, i.e. it is either a *nodal or cuspidal cubic*, not a nonsingular cubic (which would be an elliptic curve, i.e. have genus 1, not genus 0).

So, if P lies on a secant line to $\mathcal{C}$ then the image is a nodal cubic. If, on the other hand, P lies on a tangent line to $\mathcal{C}$, then the projected curve will have a cusp.

So, let's take the union of all the tangent lines to $\mathcal{C}$. These give us a surface in $\mathbb{P}^3$ (do a coarse count) which is the *tangent envelope* of $\mathcal{C}$, and none of the points of it lie on a secant line to $\mathcal{C}$. (This is clear since a plane cubic curve can have at most one singular point, so the point we project from cannot be, at the same time, on both a tangent line and on a secant line to $\mathcal{C}$). All the other points of $\mathbb{P}^3$ do lie on a secant line to $\mathcal{C}$.

So, this is an example where the union of all the secant lines gives us the complement of a surface in $\mathbb{P}^3$, which is open and not closed. So, the simple union of all the secant lines, in this case, is not a projective variety. It's closure is all of $\mathbb{P}^3$. Hence with our definition $Sec_1(\mathcal{C})$ is all of $\mathbb{P}^3$.

The first general question that I want to confront about secant varieties and joins is: How big are they? i.e. Given $\mathbb{X}_0, \ldots, \mathbb{X}_s \subset \mathbb{P}^n$,

$$\text{what is the dimension of } J(\mathbb{X}_0, \ldots, \mathbb{X}_s)?$$

There is an easy way to make an estimate. Let's suppose that $\dim \mathbb{X}_i = d_i$ and to make matters simple, let's also suppose that all the $\mathbb{X}_i$ are non-degenerate. Now, to pick a point from each $\mathbb{X}_i$ is like picking a point in $\mathbb{X}_0 \times \cdots \times \mathbb{X}_s$ (*more or less*). That's a choice of 'dimension' equal to $d_0 + d_1 + \ldots + d_s$.

Now we connect those points with a $\mathbb{P}^s$, which adds dimension s to the previous choice.

So, our first estimate is that:

$$\text{the expected dimension of } J(\mathbb{X}_0, \ldots, \mathbb{X}_s) = d_0 + \cdots + d_s + s$$

and, if all the $\mathbb{X}_i = \mathbb{X}$, then

$$\text{the expected dimension of } Sec_s(\mathbb{X}) = (s+1)\dim \mathbb{X} + s.$$

On the other hand, our varieties all lie in $\mathbb{P}^n$ and so none of the dimensions of these joins can ever exceed n. So, let's turn this informal argument into a formal definition.

Definition: Let $\mathbb{X}_i,\ i = 0, \ldots, s\ (s \leq n)$ be non-degenerate varieties in $\mathbb{P}^n$. The

$$\textit{expected dimension of } J(\mathbb{X}_0, \ldots, \mathbb{X}_s)$$

$$= \min\{s + \sum_{i=0}^{s} \dim \mathbb{X}_i,\ n\}$$

and if all the $\mathbb{X}_i = \mathbb{X}$, then

$$\textit{expected dimension of } Sec_s(\mathbb{X}) = \min\{s + (s+1)\dim \mathbb{X},\ n\}.$$

If the dimension of $Sec_s(\mathbb{X})$ is not what is expected we will say that $\mathbb{X}$ is *defective*.

> **Note:** The *expected dimension* is always greater than or equal to the actual dimension. So, in searching for the dimensions of these varieties we really are asking if the expected dimension is actually reached. To be more precise, we define

$$\delta_t(\mathbb{X}) :=$$

$$\textit{expected dimension of } Sec_t(\mathbb{X}) - \dim Sec_t(\mathbb{X})$$

and call it the t^{th}*-secant defect of* $\mathbb{X}$.

Before trying to establish some general theorems about these dimensions I would like to look at several examples in detail. The four examples I will consider are: the Segre Varieties, the Veronese Varieties, the Segre-Veronese varieties and the Varieties of Reducible Forms. There are interesting open questions about the secant varieties for all of these examples and this will be an opportunity to make some of them comprehensible.

The Segre Varieties with Two Factors, i.e. $\mathbb{P}^n \times \mathbb{P}^m$

If we use $[a_0 : \ldots : a_n]$ as coordinates in $\mathbb{P}^n$ and $[b_0 : \ldots : b_m]$ as coordinates in $\mathbb{P}^m$ then we can view $\mathbb{P}^n \times \mathbb{P}^m$ as a subvariety of $\mathbb{P}^N$, where $N = (n+1)(m+1) - 1$. It consists of all the points whose coordinates are $[\ldots : a_i b_j : \ldots]$ (obviously we have to decide on an order in which to put these products).

I prefer to think about this in the following way: $\mathbb{P}^n = \mathbb{P}(V)$, V an $(n+1)$-dimensional vector space and $\mathbb{P}^m = \mathbb{P}(W)$,

W an $(m+1)$-dimensional vector space then we can think of $\mathbb{P}^N$ as $\mathbb{P}(V \otimes W)$, and the embedding of the product described above as

$$\begin{array}{ccccc} \mathbb{P}(V) & \times & \mathbb{P}(W) & \longrightarrow & \mathbb{P}(V \otimes W) \\ [v] & \times & [w] & \longrightarrow & [v \otimes w]. \end{array}$$

To see how to obtain the coordinate description of the Segre map, we recall that if $v_0, \ldots, v_n$ and $w_0, \ldots, w_m$ are bases for V and W respectively then $v_i \otimes w_j$ is a basis for $V \otimes W$ and if t is a tensor in $V \otimes W$ then $t = \sum_{i,j} a_{i,j} v_i \otimes w_j$.

If we think of the elements of $\mathbb{P}(V \otimes W) = \mathbb{P}^N$ as equivalence classes of $(n+1) \times (m+1)$ matrices (the equivalence class containing the matrix M consists of all matrices λM, with $\lambda \neq 0$, $\lambda \in \mathbb{C}$), then the tensor t will correspond to the matrix whose (i,j) entry is $a_{i,j}$.

Thus, the coordinate form of the map I described above can be seen easily as the matrix product

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \times \begin{bmatrix} b_0 & b_1 & \cdots & b_m \end{bmatrix} \longrightarrow \begin{bmatrix} a_0 b_0 & \cdots & a_0 b_m \\ \vdots & & \vdots \\ a_n b_0 & \cdots & a_n b_m \end{bmatrix}$$

Notice that the image of $v \otimes w$ is a matrix of rank 1. Since any matrix of rank 1 can be written in this form, we see that the image of the Segre map can be identified with the $(n+1)\times(m+1)$ matrices of rank 1.

If we use $Z_{ij}, (0 \le i \le n,\ 0 \le j \le m)$ for coordinates in $\mathbb{P}^{(n+1)(m+1)-1}$ and set $S = \mathbb{C}[Z_{ij}]$, then the image of the Segre map is the set of zeroes of the ideal of S generated by the 2×2 minors of the generic matrix

$$\mathcal{Z} = \begin{bmatrix} Z_{0,0} & \cdots & Z_{0,m} \\ \vdots & & \vdots \\ Z_{n,0} & \cdots & Z_{n,m} \end{bmatrix}$$

Since we are now thinking of $\mathbb{X} = \mathbb{P}^n \times \mathbb{P}^m$ as the set of rank 1 matrices inside the set of all the matrices, how should we think about, for example, $Sec_1(\mathbb{X})$?

Two distinct points of $\mathbb{X}$ correspond to two distinct rank 1 matrices, let's call them A and B, and the secant line of X joining those points consists of all the matrices of the form $\lambda A + \mu B$. What are these matrices?

Lemma: Let M be an $(n+1)\times(m+1)$ matrix of rank r. Then M is the sum of r matrices of rank 1.

Conversely, a sum of r matrices of rank 1 is a matrix of rank $\le r$.

Proof: If M is an $(n+1)\times(m+1)$ matrix of rank r then we can write

$$M = \begin{bmatrix} | & \cdots & | \\ c_1 & \cdots & c_r \\ | & \cdots & | \end{bmatrix} \begin{bmatrix} ---- & v_1 & ---- \\ & \vdots & \\ ---- & v_r & ---- \end{bmatrix}$$

$$= \begin{bmatrix} | \\ c_1 \\ | \end{bmatrix} \begin{bmatrix} -- & v_1 & -- \end{bmatrix} + \begin{bmatrix} | \\ c_2 \\ | \end{bmatrix} \begin{bmatrix} -- & v_2 & -- \end{bmatrix} + \cdots$$

$$+ \begin{bmatrix} | \\ c_r \\ | \end{bmatrix} \begin{bmatrix} -- & v_r & -- \end{bmatrix}$$

and each of these summands is a matrix of rank 1.

Corollary: Let $\mathbb{X} = \mathbb{P}^n \times \mathbb{P}^m$, then $Sec_r(\mathbb{X}) \subset \mathbb{P}^{(n+1)(m+1)-1}$ consists of all the matrices of rank $\leq r+1$. In particular $Sec_1(\mathbb{X})$ consists of all of the matrices of rank ≤ 2, i.e. all those $(n+1)(m+1)$ matrices for which all 3×3 minors vanish.

Moreover, if $n \leq m$ then the first value of r for which $Sec_r(\mathbb{X}) = \mathbb{P}^{(n+1)(m+1)-1}$ is $r = n$.

For $\mathbb{X} = \mathbb{P}^n \times \mathbb{P}^m$ we have a very nice collection of results about their secant varieties, all of which come from the following observation:

> Let $\mathcal{Z}$ be as above: then the variety $Sec_r(\mathbb{X})$ is defined by the ideal of all the $(r+2) \times (r+2)$ minors of $\mathcal{Z}$.

The ideals of minors of generic matrices have been studied for over a century and there are many results about these ideals. For example, if we let $I_t(\mathcal{Z})$ denote the ideal generated by the $t \times t$ minors of $\mathcal{Z}$, then it is known that, in the polynomial ring

$$S = k[Z_{i,j} \ \mid \ 0 \leq i \leq n, 0 \leq j \leq m]$$

in $(n+1)(m+1)$ variables, the ideal $I_t(\mathcal{Z})$ has height $= (n+1-t+1)(m+1-t+1)$ and that the quotient ring $S/I_t(\mathcal{Z})$ is arithmetically Cohen-Macaulay.

Since $Sec_{t-2}(\mathbb{X})$ is defined by $I_t(\mathcal{Z})$ and the

$$\text{Krull dimension of } S/I_t(\mathcal{Z})$$

$$= (n+1)(m+1) - (n+1-t+1)(m+1-t+1)$$

it follows that

$$\dim Sec_{t-2}(\mathbb{X}) = \text{ Krull dimension of } S/I_t(\mathcal{Z}) - 1$$

$$= (n+m)(t-1) - (t-2)^2$$

The degree, the Hilbert function and indeed the entire minimal free resolution of $S/I_t(\mathcal{Z})$ is also known. These results are the paradigm of what one would like to know about secant varieties.

Example: Let's consider the case of $t = 3$ and $n = m = 2$. We have $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. The result above says that $Sec_1(\mathbb{P}^2 \times \mathbb{P}^2)$ has dimension $4(2) - 1 = 7$, i.e. it is a hypersurface in $\mathbb{P}^8$. Note that the *expected dimension* of $Sec_1(\mathbb{P}^2 \times \mathbb{P}^2)$ is $\min\{2 \cdot 4 + 1, 8\} = 8$. So, this variety is deficient.

It's not hard to show that **All** the proper secant varieties for $\mathbb{P}^n \times \mathbb{P}^m$ are deficient. (i.e. those which are proper subsets of the enveloping projective space).