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Betti Numbers and Generic Initial Ideals

Lecture 5

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Linear resolutions

Definition I has a linear resolution (or d-linear resolution) if it is generated in degree d and $\operatorname{reg}(I) = d$,

i.e. $\beta_{ij}(I) = 0$ for all $j \neq d + i$

i.e. the Betti diagram is just one line.

Example $I = (x_1x_3 - x_2^2, x_1x_4 - x_2x_3, x_2x_4 - x_3^2)$ has 2-linear resolution.

$$0 \longrightarrow R(-3)^2 \longrightarrow R(-2)^3 \longrightarrow I$$

```
BettiDiagram(I);
        0    1
---------------
 2:     3    2
---------------
Tot:    3    2
--------------------------------
```

Remark If I and J have linear resolutions and the same HF, they have the same Betti numbers.

Remark If I has a linear resolution then $\beta_{ij}(I) = \beta_{ij}(\mathrm{gin}_{\mathrm{rlex}}(I))$.

Definition I homogeneous, $d \in \mathbb{N}$. Define $I_{\langle d \rangle}$ to be the ideal generated by the elements of degree d in I.

Example $I = (x^2 + y^2, x^3, z^4)$ in $K[x, y, z]$. Then $I_{\langle 3 \rangle}$ is $I = (x^2 + y^2)(x, y, z) + (x^3)$.

Definition I is componentwise linear if for every d the ideal $I_{\langle d \rangle}$ has a linear resolution.

Lemma I is componentwise linear iff $I_{\langle d \rangle}$ has a linear resolution for those d such that I has minimal generators of degree d.

Corollary I has a linear resolution $\Rightarrow$ I is componentwise linear.

Lemma If I is stable (e.g. strongly stable) then I is componentwise linear.

Proof: $I_{\langle d \rangle}$ is stable + EK.

Definition A vector space V of forms of degree i is said to be Gotzmann if $\dim VR_1$ is the smallest possible,

i.e. $\dim VR_1 = \dim LR_1$ where $L = \mathrm{Lex}(V)$,

i.e. $\mathrm{Lex}(VR_1) = R_1 \,\mathrm{Lex}(V)$.

An ideal I Gotzmann if I_i is Gotzmann for all i.

Gotzmann Persistence: V Gotzmann $\Rightarrow VR_1$ Gotzmann.

Lemma I is Gotzmann $\Rightarrow I$ is componentwise linear.

Proof: Gotzmann Persistence Thm.+ Bigatti-Hulett-Pardue Thm.

Theorem (Aramova-Herzog-Hibi) $\mathrm{char}\, 0$, $J = \mathrm{gin}_{\mathrm{rlex}}(I)$. TFAE

1) I is componentwise linear

2) $\beta_{ij}(I) = \beta_{ij}(J)$ all i, j

3) $\beta_{0j}(I) = \beta_{0j}(J)$ all j

4) In generic coordinates, I is minimally generated by a revlex Gröbner basis.

$1) \Rightarrow 2)$ true for all char.

$3) \Rightarrow 1)$ needs $\mathrm{char}\, 0$: $I = (x^p, y^p)$ does not have a linear resolution and $I = \mathrm{gin}_{\mathrm{rlex}}(I)$ in $\mathrm{char}\, p$.

Crucial fact: the crystallization principle for gins.

Crystallization Principle

Theorem (Crystallization) In $\operatorname{char} 0$, if I is generated in degree $\leq d$ and $\operatorname{gin}_\tau(I)$ has no generators in degree $d+1$, then $\operatorname{gin}_\tau(I)$ has no generators in degree $> d$ i.e. $\operatorname{reg}(I) \leq d$.

Proof: Replace I with $I_{\langle d \rangle}$, $\operatorname{gin}_\tau(I)$ is strongly stable, hence it has linear syzygies$+$ Buchberger algorithm.

```
Use R::=Q[x[1..4]];
I:=Ideal(x[1],x[2])*Ideal(x[3],x[4])
*Ideal(x[1],x[3]);

BettiDiagram(I);
------------------------------
         0    1    2    3
------------------------
 3:      8   12    6    1
------------------------
Tot:     8   12    6    1
------------------------------
G:=Gin(I); BettiDiagram(G);
------------------------------
         0    1    2    3
------------------------
 3:      8   12    6    1
------------------------
Tot:     8   12    6    1
------------------------------
```

```
Use R::=Q[x[1..4]], Lex;
I:=Ideal(x[1],x[2])*Ideal(x[3],x[4])
*Ideal(x[1],x[3]);

BettiDiagram(I);
-------------------------------
         0    1    2    3
-------------------------
 3:      8   12    6    1
-------------------------
Tot:     8   12    6    1
-------------------------------
G:=Gin(I);  BettiDiagram(G);
         0    1    2    3
-------------------------
 3:      8   13    8    2
 4:      1    2    1    -
-------------------------
Tot:     9   15    9    2
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```

Generalizing, we have proved the following "rigidity" behaviour:

Theorem (–Herzog-Hibi) $\mathrm{char}\, 0$, $J = \mathrm{gin}_{\mathrm{rlex}}(I)$ and k a number. TFAE

1) $\beta_{ij}(I) = \beta_{ij}(J)$ all j and all $i \geq k$

2) $\beta_{kj}(I) = \beta_{kj}(J)$ all j

New ingredient of the proof: generic Koszul homology. Koszul homology wrt generic sequences of linear forms.

If $y_1, \ldots, y_n$ are general linear forms and $1 \leq t \leq n$, set

$H_i(t, R/I) =$ i-th homology of Koszul complex associated to $y_1, \ldots, y_t$ over R/I

Key point: transfer of annihilation, that is, if for a given i and all t one has $\mathbf{m}H_i(t, R/I) = 0$ then $\mathbf{m}H_{i+1}(t, R/I) = 0$ for all t.

Description of extremal behaviour in Bigatti-Hulett-Pardue THM:

Theorem (Herzog-Hibi) $L = \mathrm{Lex}(I)$. TFAE

1) I is Gotzmann

2) $\beta_{0j}(I) = \beta_{0j}(L)$ all j

3) $\beta_{ij}(I) = \beta_{ij}(L)$ all i, j

We have rigidity wrt to $\mathrm{Lex}(I)$ and any other gins:

Theorem (–Herzog-Hibi) $\mathrm{char}\,0$, $J = \mathrm{gin}_\tau(I)$ or $J = \mathrm{Lex}(I)$ and k a number. TFAE

1) $\beta_{ij}(I) = \beta_{ij}(J)$ all j and all $i \geq k$

2) $\beta_{kj}(I) = \beta_{kj}(J)$ all j

Proof: Rigidity vs gin-revlex$+$ EK

Polarizzations and Distractions

Pardue's proof of the extremality of Lex wrt Betti numbers in arbitrary characteristic is based on polarizzations and distractions.

$R = K[x_1, \dots, x_n]$.

Distractions: $L = (L_{ij})$ a $n \times \mathbb{Z}$ matrix with L_{ij} linear forms

$$i = 1, \dots, n \quad \text{and} \quad j \in \mathbb{Z}$$

Take a monomial $m = x_1^{a_1} \dots x_n^{a_n}$

$$D_L(m) = \prod_{i=1}^{n} \prod_{j=1}^{a_i} L_{ij}$$

$$D_L(x_1^2 x_2 x_3^3) = L_{1,1} L_{1,2} L_{2,1} L_{3,1} L_{3,2} L_{3,3}$$

$$D_L : R \longrightarrow R$$

K-linear map, but not a K-algebra homomorpism.

Definition D_L is a distraction if D_L is bijective

Remark D_L is a distraction iff elements in different rows of L are linearly independent, i.e.

$$< L_{1,j_1}, L_{2,j_2}, \ldots, L_{n,j_n} >= R_1$$

for all $j_1, \ldots, j_n \in \mathbb{Z}$.

Example 1) $L_{ij} = x_i$ (trivial)

2) $L_{ij} = x_i + \sum_{k>i} *x_k$ (triangular)

Theorem L distraction and I monomial ideal then

1) $D_L(I)$ is an ideal

2) I and $D_L(I)$ have the same HF and Betti numbers

3) If F is any $\mathbb{Z}^n$-graded free resolution of I then $D_L(F)$ is a graded free resolution of $D_L(I)$.

Proof: 1) the key point is that

$$D_L(mR_1) = D_L(m)R_1$$

for every monomial m.

2) That I and $D_L(I)$ have the same HF follows because D_L is an isomorphism of vector spaces. That they have the same Betti numbers follows from 3) applied to a minimal free resolution.

3) One extends the action of D_L to multigraded free modules and maps and shows that the

resulting D_L acts as a functor from $\mathbb{Z}^n$-graded objects to $\mathbb{Z}$-graded objects which preserves HF of homology modules and so exactness.

The saturation of I is

$$I^{\mathrm{sat}} = \{f \in R : fx_i^k \in I \ \forall \ i \text{ and } k >> 0\}$$

I is saturated iff $I = I^{\mathrm{sat}}$, equivalently R/I has at least a non-invertible homogeneous nzd.

An important observation:

Lemma If I is a monomial saturated ideal and L is "generic enough" then $D_L(I)$ is radical.

"generic enough": for every $k < n$ and for every $1 \leq i_1 < \cdots < i_k \leq n$ the linear spaces $\langle L_{i_1 j_1}, L_{i_2 j_2}, \ldots, L_{i_k j_k} \rangle$ and $\langle L_{i_1 v_1}, L_{i_2 v_2}, \ldots, L_{i_k v_k} \rangle$ are distinct if $(j_1, \ldots, j_k) \neq (v_1, \ldots, v_k)$.

Proof: D_L commutes with taking intersection. Enough to deal with irreducible ideals.

Theorem (Bigatti-C-Robbiano) If I is strongly stable monomial ideal then

$$\mathrm{gin}_{\mathrm{rlex}}(D_L(I)) = I$$

for every distraction D_L.

Corollary Every saturated strongly stable ideal I is the Gin-revlex of a reduced ideal

Corollary If I is strongly stable and P is its polarizzation (a square-free monomial ideal in many more variables) then

$$\mathrm{gin}(P) = I$$

Example $I = (x_1^2, x_1x_2, x_2^3)$

$P = (x_{11}x_{12}, x_{11}x_{21}, x_{21}x_{22}x_{23})$
$\mathrm{gin}(P) = I$ after $x_{ij} \longrightarrow x_i$.

In general $\mathrm{gin}_\tau(D_L(I)) \neq I$ if I is strongly stable and τ is not revlex.

Pardue: if L is generic then

$$\mathrm{gin}_{Lex}(D_L(I)) = I$$

if and only if

I is a lex-segment ideal

One can ask

$$\mathrm{gin}(D_L(I)) = \mathrm{gin}(I) \quad ????$$

for a general monomial ideal.

One has

Lemma If I is componentwise linear (Gotzmann) then $D_L(I)$ is componentwise linear (Gotzmann).

Corollary If I is componentwise linear (e.g. stable) then $\mathrm{gin}(I)$ and $\mathrm{gin}(D_L(I))$ have the same Betti numbers

Surprise : there exist stable monomial ideals I and a distractions L such that

$$\text{gin}(D_L(I)) \neq \text{gin}(I)$$

$$G := [x_3^2 x_4^2, x_2^3];$$

I:=Stable(G);

the smallest stable ideal containing G.

J:=GenericDistraction(I);

Gin(I)=Gin(J);

FALSE ($x_1 x_3^2 x_4$ in GinI and not in GinJ and $x_2^2 x_3 x_4$ in GinJ and not in GinI)

But:

Theorem If m is a monomial and $I = \mathrm{Stable}(m)$ then

$$\mathrm{gin}(D_L(I)) = \mathrm{gin}(I)$$

for all L.

Gin, in and reduction numbers

Gin and Fröberg conjecture

Gin of complete intersections

Gin Lex

Gin and shifting