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Betti Numbers and Generic Initial Ideals

Lecture 4

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I homogeneous ideal of $R = K[x_1, \ldots, x_n]$

$\mathbf{m} = (x_1, \ldots, x_n)$.

Hilbert function and series

$$HF(R/I, i) \quad i \in \mathbb{N} \longrightarrow \dim_K [R/I]_i$$

$$HS(R/I, z) = \sum \dim_K [R/I]_i z^i$$

$$HS(R/I, z) = \frac{h(z)}{(1-z)^d}$$

$h(1) \neq 0$ and $d =$Krull dimension of R/I.

$h(1)$ is called degree or multiplicity of R/I.

Betti numbers and minimal free resolutions

$I = (f_1, \ldots, f_k)$ minimal generators we get a surjective R-module homomorphism

$$\psi : R^k \longrightarrow I \text{ defined by } e_i \longrightarrow f_i$$

$$\mathrm{Syz}_1(I) = \mathrm{Ker}\,\psi \text{ — first syzygies}$$

Taking minimal generators of $\mathrm{Syz}_1(I)$ we can iterate the procedure.

By Hilbert syzygy theorem, after a finite number of step we get a trivial kernel. We obtain a minimal free resolution. It is unique up to isomorphism of complexes.

$\beta_0(I)$=number of minimal generators of $I = \dim I/\mathbf{m}I$

$\beta_{0j}(I)$=number of minimal generators of degree j
$= \dim I_j/R_1 I_{j-1}$

$\beta_i(I)$=number of minimal i-syzygies of I

$\beta_{ij}(I)$ =number of minimal i-syzygies of I of degree j

$$0 \longrightarrow \oplus R(-j)^{\beta_{pj}} \longrightarrow \ldots \longrightarrow \oplus R(-j)^{\beta_{0j}} \longrightarrow I$$

Projective dimension (=length of the resolution)

$$\mathrm{proj.dim}(I) = \max\{i : \beta_{ij}(I) \neq 0 \text{ some } j\}$$

Castelnuovo-Mumford regularity (=the width of the resolution)

$$\mathrm{reg}(I) = \max\{j - i : \beta_{ij}(I) \neq 0\}$$

After Krull dimension and degree, the Castelnuovo Mumford regularity is the most import invariant of I as it bounds simultaneously

degrees of syzygies of I,

vanishing of local cohomology modules $H^i_{\mathbf{m}}(R/I)$ of R/I,

the gratest integer for which HFunction and HPolynomial of R/I do not agree

degree of elements in revlex Gröbner bases I in generic coordinates.

Example $I = (x^2, xy, xz, y^3)$

$$\beta_{02}(I) = 3, \quad \beta_{03}(I) = 1$$

$\psi_1 : R^4 \longrightarrow I$ defined by $\begin{array}{ccc} e_1 & \longrightarrow & x^2 \\ e_2 & \longrightarrow & xy \\ e_3 & \longrightarrow & xz \\ e_4 & \longrightarrow & y^3 \end{array}$

$\mathrm{Syz}_1(I)$ is generated by $\begin{array}{l} s_1 = ye_1 - xe_2 \\ s_2 = ze_1 - xe_3 \\ s_3 = ze_2 - ye_3 \\ s_4 = y^2e_2 - xe_4 \end{array}$

e_i gets the degree of its target. So, for instance, s_1 has degree 3.

$$\beta_{13}(I) = 3, \quad \beta_{14}(I) = 1$$

$\psi_2 : R^4 \longrightarrow \mathrm{Syz}_1(I)$ sending $e'_i \longrightarrow s_i$

$\mathrm{Syz}_2(I)$ is generated by $ze'_1 - ye'_2 + xe'_3$.

$$\beta_{24}(I) = 1$$

$$0 \longrightarrow (-4) \longrightarrow \begin{matrix} (-3)^3 \\ \oplus \\ (-4) \end{matrix} \longrightarrow \begin{matrix} (-2)^3 \\ \oplus \\ (-3) \end{matrix} \longrightarrow I$$

Betti Diagram:

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BettiDiagram(I);
        0    1    2
--------------------
 2:     3    3    1
 3:     1    1    -
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Tot:    4    4    1
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$$\text{proj.dim}(I) = 2$$

$$\text{reg}(I) = 3$$

Remark The Betti numbers of I determine the HF of I. If β_{ij} are the Betti numbers of I then the Hilbert series of R/I is given by

$$HS(R/I, z) = \frac{1 + \sum_{ij} (-1)^{i+1} \beta_{ij} z^j}{(1-z)^n}$$

Not the other way round: two ideals can have the same HF but different Betti numbers. $I = (x^2, y^2)$ and (x^2, xy, y^3) have both HF $(1, 2, 1, 0)$ and different number of generators.

Definition I has a pure resolution if for all i the minimal i-syzygies (if any) have all the same degree, that is for all i there exits at most one j so that $\beta_{ij}(I) \neq 0$.

If I, J have the same HF and both have pure resolution then they have the same Betti numbers.

Definition I has a linear resolution if it is generated in one degree, say d, and $\beta_{ij}(I) = 0$ for all $j \neq i + d$.

Problem Consider an homogeneous ideal I in $K[x, y, z, t]$. Compute with CoCoA its regularity, say it is r. Set $I_{\geq k} = I \cap \mathbf{m}^k$. Compute the resolution of $I_{\geq k}$ for some values $k < r$ and some values $k \geq r$. Guess what it is going on.

Invariants and deformations

τ term order $\longrightarrow$ initial ideal $\mathrm{in}_\tau(I)$

I and $\mathrm{in}_\tau(I)$ have the same Hilbert function

$$\beta_{ij}(I) \le \beta_{ij}(\mathrm{in}_\tau(I))$$

$$\mathrm{proj.dim}(I) \le \mathrm{proj.dim}(\mathrm{in}_\tau(I))$$

$$\mathrm{reg}(I) \le \mathrm{reg}(\mathrm{in}_\tau(I))$$

(usually $<$)

Problem Check with CoCoA in some examples.

Gin-revlex

For ideals I and J one defines

$I : J = \{f \in R : fJ \subseteq I\}$

Note: f is a n.z.d. mod I iff $I : (f) = I$

The revlex order has some peculiar properties, for instance:

*) $\mathrm{in}(I : (x_n)) = \mathrm{in}(I) : x_n$

*) $\mathrm{in}(I + (x_n)) = \mathrm{in}(I) + (x_n)$

*) $x_j, \dots, x_n$ form a regular sequence mod I iff $x_j, \dots, x_n$ form a regular sequence mod $\mathrm{in}_{\mathrm{rlex}}(I)$

For a monomial m denote

$$\max(m) = \max\{i \ : \ x_i | m\}$$

These properties play the key role in the proof of:

Theorem (Bayer-Stillman) $J = \mathrm{gin}_{\mathrm{rlex}}(I)$

$$\mathrm{proj.dim}(I) = \mathrm{proj.dim}(J)$$

$$\mathrm{reg}(I) = \mathrm{reg}(J)$$

and if $J = (m_1, \ldots, m_k)$ then

$$\mathrm{proj.dim}(J) = \max_i\{\max(m_i)\} - 1$$

Recent works Caviglia-Sbarra, Caviglia and Bermejo-Gimenez show that THM holds also by replacing $\mathrm{gin}_{\mathrm{rlex}}(I)$ with $\mathrm{in}_{\mathrm{rlex}}(I)$ provided that $\mathrm{in}_{\mathrm{rlex}}(I)$ has only associated prime ideals of type $(x_1, x_2 \ldots, x_k)$.

In $\mathrm{char}\,0$, the regularity of a Borel-fixed(=strongly stable) ideal J is $\max\{\deg(m_i)\}$

Corollary In $\mathrm{char}\,0$, let $\mathrm{gin}_{\mathrm{rlex}}(I) = (m_1, \ldots, m_k)$ then

$$\mathrm{proj.dim}(I) = \max\{\max(m_i)\} - 1$$

$$\mathrm{reg}(I) = \max\{\deg(m_i)\}$$

Syzygies of (strongly) stable ideals

J strongly stable ideal with generators $m_1, \ldots, m_k$. Order the m_i so that $\deg(m_i) < \deg(m_{i+1})$ or $\deg(m_i) = \deg(m_{i+1})$ and $m_i > m_{i+1}$ revlex.

For every m_v with $i = \max(m_v)$ and for every $j < i$ we have $(x_j/x_i)m_v \in J$ i.e. there exists w and n such that

$$(x_j/x_i)m_v = nm_w$$

Since either $\deg m_w < \deg m_v$ or ($m_w > m_v$ revlex) then $w < v$.

$$x_j m_v = x_i n m_w$$

This is a syzygy of the m_i's.

$\psi : R^k \longrightarrow J \quad \psi(e_i) = m_i$

$t(v, j) := x_j e_v - x_i n e_w \in \mathrm{Syz}_1(J)$

Indeed,

Proposition The syzygies $t(v, j)$ with $v > 1$, $j < \max(m_v)$ minimally generate $\mathrm{Syz}_1(J)$.

Sketch: The binomial syzygies $s(i, j) = ae_i - be_j$ with $\mathrm{GCD}(a, b) = 1$ and $i > j$ generates $\mathrm{Syz}_1(J)$.

Enough to prove the $s(i, j)$ are combinations of the $t(v, j)$. Induction on i, j, $s(2, 1)$ is $t(2, 1)$.

If there is a variable x_k in a with $k < \max(m_i)$ then substracting from $s(i, j)$ a multiple of $t(i, k)$ we get a multiple of $s(w, j)$ or $s(j, w)$ with $w < i$....done by induction.

If all the variables x_k in a have $k \geq \max(m_i)$ then either $\deg m_i = \deg m_j$ and $m_j > m_i$ or $\deg m_j < \deg m_i$ and m_i not a minimal generator. Contradiction.

It follows that if J is strongly stable minimally generated by $m_1, \ldots, m_k$ then:

$$\beta_1(J) = \sum_v (\max(m_v) - 1)$$

$$\beta_{1j}(J) = \sum_{*}(\max(m_v) - 1)$$

where $\sum_*$ is sum over the v with $\deg(m_v) = j - 1$.

One can generalize in two ways: larger class and full resolution!!!

Eliahou-Kervaire resolution

Definitin A monomial ideal J is stable if for all $m \in J$ and $j < i = \max(m)$ then $(x_j/x_i)m \in J$. (enough test generators).

For a set of monomials A we put:

$$M_i(A) = |\{m \in A : \max(m) = i\}|$$
$$M_{\leq i}(A) = |\{u \in A : \max(u) \leq i\}|$$

$$M_{ij}(A) = |\{m \in A : \begin{array}{l} \max(m) = i \text{ and} \\ \deg(m) = j \end{array} \}|$$

If J is an ideal or a vector space minimally generated by a set of monomials A we set

$$M_i(J) = M_i(A) \text{ and } M_{ij}(J) = M_{ij}(A)$$

Theorem (Eliahou-Kervaire) J stable monomial ideal. Then

$$\beta_i(J) = \sum_{s=i+1}^{n} M_s(J)\binom{s-1}{i}$$

$$\beta_{ij}(J) = \sum_{s=i+1}^{n} M_{s,j-i}(I)\binom{s-1}{i}$$

We have the B-lemmata:

LEMMA Let I, J be strongly stable ideals with the same Hilbert function. If $M_{\leq i}(J_j) \leq M_{\leq i}(I_j)$ for all i, j then

i) $M_i(I) \leq M_i(J)$ for all i.

ii) $\beta_{ij}(I) \leq \beta_{ij}(J)$ for all i, j.

LEMMA (Bigatti, Bayer) Let I be a strongly stable ideal and $L = \mathrm{Lex}(I)$ be the corresponding lex-segment. Then $M_{\leq i}(L_j) \leq M_{\leq i}(I_j)$ for all i, j.

Extremality of Lex and Betti numbers

In the class of the ideals with a given HF the Lex-segment has the largest Betti numbers.

Theorem (Bigatti, Hulett, Pardue) I homogeneous ideal, $L = \mathrm{Lex}(I)$ corresponding lex-segment. Then

$$\beta_{ij}(I) \leq \beta_{ij}(L)$$

Proved in $\mathrm{char}\, 0$ by Bigatti and Hulett. Extended to arbitrary char by K.Pardue via polarizations-distractions.

Proof in $\mathrm{char}\, 0$: Replacing I with $\mathrm{gin}(I)$ we may assume that I is strongly stable. Then use EK and the B-lemmata.

Among the "gins", the gin-revlex gives the best upper approximation of the Betti numbers

Theorem (-) In $\mathrm{char}\, 0$: If τ is any t.o. and $G = \mathrm{gin}_{\mathrm{revlex}}(I)$

$$\beta_{ij}(I) \leq \beta_{ij}(G) \leq \beta_{ij}(\mathrm{gin}_\tau(I))$$

One could ask whether, among the gins of I, the gin-lex has the largest Betti numbers. Or whether among the gins of I there exists an ideal whose Betti numbers bounds above the Betti numbers of any other gins. The answer is negative in both cases.

Relevant examples are constructed by using almost Borel fixed ideals.

Let A be a strongly stable vector space of monomial of degree i.

A lower neighbor of A is a monomial m not in A but moved into A by any Borel move.

Let W be the vector space generated by the lower neighbors of A.

Let $V \subseteq W$ be a subspace.

The vector space $A + V$ is called an almost Borel fixed space.

A homogeneous ideal I is said to be almost Borel-fixed if for each $d \in \mathbb{N}$ the space I_d is almost Borel-fixed.

The main property of almost Borel-fixed spaces and ideals

For every term order τ one has:

$$\operatorname{gin}_\tau(A+V) = A + \operatorname{in}_\tau(V)$$

Example The simplest almost Borel-fixed space (which is not Borel-fixed) is the following: in 3 variables,

$A = \langle x_1^2, x_1x_2 \rangle$, lower neighbors are $\{x_1x_3, x_2^2\}$ $V = \langle x_1x_3 + x_2^2 \rangle$.

Then the almost Borel-fixed space $A+V$ has only two distinct gins, the gin-revlex $A + \langle x_2^2 \rangle$ and the gin-lex $A + \langle x_1x_3 \rangle$.

When does it happen that $\beta_{ij}(I) = \beta_{ij}(\mathrm{gin}_{\mathrm{rlex}}(I))$?

For special values of i, j it is an equality.

Definition A Betti number $\beta_{ij}(I)$ is extremal if $\beta_{hk}(I) = 0$ for all $h \geq i$ and $k \geq j + 1$.

Theorem (Bayer-Charalambous-Popescu) The positions and the values of the extremal Betti numbers of I and $\mathrm{gin}_{\mathrm{rlex}}(I)$ are the same.

Example

I
$(x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2^2, x_3x_5)$

gin-revlex
$(x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_1x_4, x_3^3)$

gin-lex
$(x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2^2, x_2x_3x_4, x_2x_3^2,$
$x_2x_3x_5, x_2x_4^3, x_3^4)$

lex-segment
$(x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2^2, x_2x_3^2, x_2x_3x_4,$
$x_2x_3x_5, x_2x_4^3, x_2x_4^2x_5, x_2x_4x_5^3, x_2x_5^4, x_3^5x_4, x_3^6,$
$x_3^5x_5, x_3^4x_4^3)$

I

2|	6	8	4	1
3|	0	1	1	0

gin-revlex

2|	6	9	5	1
3|	1	2	1	0

gin-lex

2|	6	11	10	5	1
3|	3	9	10	5	1
4|	2	5	4	1	0

lex-segment

2|	6	11	10	5	1
3|	3	9	10	5	1
4|	2	7	9	5	1
5|	2	8	12	8	2
6|	3	9	10	5	1
7|	1	3	3	1	0