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Betti Numbers and Generic Initial Ideals

Lecture 3

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I homogeneous ideal of $R = K[x_1, \ldots, x_n]$

Hilbert function and polynomial of R/I and of I

$$HF(R/I, i) \quad i \in \mathbb{N} \longrightarrow \dim_K [R/I]_i$$

$$HF(I, i) \quad i \in \mathbb{N} \longrightarrow \dim_K I_i$$

$$HF(R/I, i) + HF(I, i) = \binom{n-1+i}{n-1}$$

$HF(R/I, i)$ agrees for $i >> 0$ with a polynomial, Hilbert polynomial of R/I, whose degree is one less than the Krull dimension of R/I.

I and $\mathrm{in}_\tau(I)$ have the same Hilbert function.

I and $\mathrm{gin}_\tau(I)$ have the same Hilbert function.

Segments of monomials

Let τ be t.o. on $R = K[x_1, \ldots, x_n]$.

Assume that $x_1 > \cdots > x_n$.

V be a vector space generated by monomials of degree i.

Definition V is a τ-segment if whenever m_1, m_2 are monomials of degree i such that $m_1 > m_2$ and $m_2 \in V$ then also $m_1 \in V$.

Given τ, i and

$$d \leq \dim R_i = \binom{n-1+i}{n-1}$$

there exists exactly one τ-segment of dimension d and degree i: it is vector space generated by the d largest monomials of degree i.

Denote it by $\mathrm{Seg}_\tau^n(i,d)$ or just by $\mathrm{Seg}_\tau(i,d)$ if n is clear from the context.

Example If $n = 3$ then

$$\mathrm{Seg}^3_{\mathrm{lex}}(2,4) = \mathrm{Seg}^3_{\mathrm{rlex}}(2,4) = \langle x_1^2, x_1x_2, x_2^2, x_1x_3 \rangle$$

If $n = 4$ then

$$\mathrm{Seg}^4_{\mathrm{lex}}(2,4) = \langle x_1^2, x_1x_2, x_1x_3, x_1x_4 \rangle$$

$$\mathrm{Seg}^4_{\mathrm{rlex}}(2,4) = \langle x_1^2, x_1x_2, x_2^2, x_1x_3 \rangle$$

Remark $\mathrm{Seg}^n_{\mathrm{rlex}}(i,d)$ is independent of n.

Definition A monomial ideal I is a τ-segment if every homogeneous component I_i of I is a τ-segment, equivalently ,
if m_1, m_2 are monomials of the same degree and $m_1 > m_2 \in I$ then $m_1 \in I$.

Warning: Not enough to test the condition for ideal-generators:

Example (x_1) is a revlex-segment in degree 1 but not in degree 2 since $x_2^2 > x_1 x_3 \in I$ and $x_2^2 \notin I$.

But it is enough to test ideal-generators to check whether an ideal is a lex-segment, i.e.

A monomial ideal I is a lex-segment if whenever m_1, m_2 are monomials of the same degree and $m_1 > m_2$ and m_2 is a generator of I then $m_1 \in I$.

The gins (generic initial ideals) "tend" to be segments since the maximal potential support for a vector space of dimension d of forms of degree i "is" the correspondent τ-segment.

But in general they are not. An obstruction comes from Hilbert functions.

A monomial τ-segment I is determined by its Hilbert function,

$$I = \oplus_i \operatorname{Seg}_\tau^n(i, \dim I_i)$$

Given a function $h : \mathbb{N} \longrightarrow \mathbb{N}$ we say that h supports a τ-segment if h is the Hilbert function of R/I where I is a τ-segment. This is equivalent to the following conditions
1) $h(0) = 1$, $h(1) = n$,
2) $h_i^* \geq 0$,
3) $\oplus_i \operatorname{Seg}_\tau^n(i, h_i^*)$ is an ideal

where

$$h_i^* = \binom{n-1+i}{n-1} - h(i)$$

Certain rings have Hilbert functions which do not support τ-segment ideals, for instance:

Lemma A function of $h : \mathbb{N} \longrightarrow \mathbb{N}$ with $h(0) = 1$ and $h(1) = n$ supports a revlex-segment ideal
iff
$h(i+1) \leq h(i)$ for all $i \geq \min\{j : h(j) < \binom{n-1+j}{n-1}\}$

Prove the only if part.

It follows that the Hilbert functions of proper $(^*)$ quotients of R with Krull dimension > 1 do not support revlex segments.

In particular, if $I \neq 0$, does not contain linear forms and R/I has Krull dimension > 1 then $\mathrm{gin}_{\mathrm{rlex}}(I)$ is NOT a revlex-segment.

The Lemma can be used also for some 0-dimensional ring:

Example $I = (x^5, y^5, z^5)$, $h =$ HF of R/I then $h(5) = 18$ and $h(6) = 19$. So $\mathrm{gin}_{\mathrm{rlex}}(I)$ is NOT a revlex-segment simply because there are no revlex-segment ideals with that HF.

But ALL Hilbert functions support lex-segment ideals. This is (a possible formulation) of Macaulay characterization of HF:

Theorem (Macaulay): A function $h : \mathbb{N} \longrightarrow \mathbb{N}$ with $h(0) = 1$ and $h(1) = n$ is the Hilbert function of a quotient of R iff $\oplus_i \mathrm{Seg}^n_{\mathrm{lex}}(i, h^*_i)$ is an ideal of R .

$$h^*_i = \binom{n-1+i}{n-1} - h(i)$$

If $h : \mathbb{N} \longrightarrow \mathbb{N}$ is a HF, then the ideal $\oplus_i \mathrm{Seg}^n_{\mathrm{lex}}(i, h^*_i)$ is denoted by $\mathrm{Lex}(h)$. It is called the lex-segment associated with h.

If I is an ideal and h is the Hilbert function of R/I then $\mathrm{Lex}(h)$ is denoted also by $\mathrm{Lex}(I)$.

An essentially equivalent formulation of Macaulay Theorem is:

For every vector space $V \subset R_i$ of dimension d set $L = \mathrm{Seg}^n_{\mathrm{lex}}(i,d)$. Then one has

$$\dim VR_1 \geq \dim LR_1$$

The vector spaces V satisfying equality deserve a special name:

Definition Let V be a vector space $V \subset R_i$ with $\dim V = d$ and set $L = \mathrm{Seg}^n_{\mathrm{lex}}(i,d)$. Then V is called Gotzmann if $\dim VR_1 = \dim LR_1$.

Problem Describe some Gotzmann spaces for $n = 3$, $i = 2$ and $d = 4$. Let $L = \mathrm{Seg}^3_{\mathrm{lex}}(2,4)$, describe all the Gotzmann spaces V with $\mathrm{in}_{\mathrm{lex}}(V) = L$.

Problem Given an Artinian function $h : \mathbb{N} \longrightarrow \mathbb{N}$ (that is $h(i) = 0$ for $i >> 0$) and a term order τ write a Cocoa function to check whether h supports a τ-segment.

Example

$I = (x^2, y^3, z^4) \subset R = K[x,y,z]$, the HF of R/I is $h = (1,3,5,6,5,3,1,0)$. So the HF of I is $h^* = (0,0,1,4,10,18,27,36,...)$

Then $\mathrm{Lex}(I)$ is generated by the 1 largest monomials in degree 2, by the 4 largest monomials of degree 3, and so on.

x^2	1
$\ldots x^2z, xy^2$	4
$\ldots, xy^2z, xyz^2, xz^3$	10
$\ldots, xz^4, y^5, y^4z, y^3z^2$	18
$\ldots, y^3z^3, y^2z^4, yz^5$	27
$\ldots, yz^6, z^7$	36 all the monomials

So $\mathrm{Lex}(I) = \mathrm{Lex}(h)$ is generated by the 10 monomials.

$$(x^2, xy^2, xyz^2, xz^3, y^5, y^4z, y^3z^2, y^2z^4, yz^5, z^7)$$

How do Borel-fixed ideals look like?

An ideal I is Borel-fixed iff it is monomial and verifies the following condition

for every monomial $m \in I$
for every $1 \leq j < i \leq n$
let t be the exponent of x_i in m.

Then $(x_j/x_i)^r m \in I$ for all $r = 1, \ldots, t$ such $\binom{t}{r} \neq 0$ in K.

Enough to test with m ideal-generators of I.

Key point: invertible diagonal matrices + elementary upper triangular matrices generate the Borel-group B_n.

Elementary upper triangular matrices $E_{ji}(a)$ with $j < i$ correspond to automorphisms:
$x_k \longrightarrow x_k$ for $k \neq i$
$x_i \longrightarrow x_i + a x_j$

Strongly stable ideals

If $\operatorname{char} 0$ then $\binom{t}{r} \neq 0$.

Definition A monomial ideal I is strongly stable if whenever $mx_i \in I$ for some monomial m then $mx_j \in I$ for every $j < i$.

Equivalently, $I : x_i = I : (x_1, \ldots, x_i)$ for every i.

Strongly stable $\Rightarrow$ Borel-fixed.

In $\operatorname{char} 0$, Borel-fixed is equivalent to strongly stable.

(x_1^p, x_2^p) Borex-fixed in $\operatorname{char} p$, not strongly stable.

$(x_1^2, x_1x_2, x_2^2, x_1x_3^2, x_3^3)$ Borex-fixed in $\operatorname{char} 3$, not strongly stable.

Sums, products, intersections and colon ideals of Borel-fixed ideals are Borel-fixed.

*) Segments are strongly stable since $x_jm > x_im$ if $j < i$.

*) In $K[x_1, x_2]$ strongly stable ideals are segments (lex-segments)

*) For ≥ 3 variables strongly stable ideals are, in general, not segments:

Example

The ideal $(x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^3, x_2^4)$ is strongly stable and not a segment.

If, by contradiction, there exists a t.o. τ such that it is a segment, then since $x_1^2x_3$ is in and $x_1x_2^2$ is out, we have $x_1^2x_3 >_\tau x_1x_2^2$.

It follows $x_1x_3 >_\tau x_2^2$ and $x_1x_2^2x_3 >_\tau x_2^4$. But x_2^4 is in and $x_1x_2^2x_3$ is out: a contradiction.

Summing up,

$$\begin{array}{ccccc} & & & & \text{gins} \\ & & & & \Downarrow \\ \text{segments} & \Rightarrow & \text{st.stable} & \Rightarrow & \text{B.fixed} \\ & \not\Leftarrow & & \not\Leftarrow & \\ & \Leftarrow & \text{if } n=2 & & \\ & & & \Leftarrow & \text{if } \operatorname{char} 0 \end{array}$$

One can introduce the Borel partial order on the set of monomials of given degree.

A Borel move: in a monomial m replace a variable x_i with x_j with $j < i$. For example, $x_1x_2^3x_3x_4^2 \longrightarrow x_1x_2^4x_4^2$.

Given monomials of same degree we say $m_1 > m_2$ in the Borel order if we can pass fro m_2 to m_1 with a series of Borel moves.

If $m_1 = x^a$ and $m_2 = x^b$, $a, b \in \mathbb{N}^n$ then $m_1 > m_2$ in the Borel order iff $a_1 \geq b_1$ and $a_1 + a_2 \geq b_1 + b_2$ and $\dots$.

The Borel order is a partial order on the set of monomials of degree i.

Let x^a and x^b be monomials of degree i. Then
$x^a > x^b$ in the Borel order
iff
$x^a >_\tau x^b$ for all term orders τ with $x_1 > \cdots > x_n$.

A strongly stable ideal I is one satisfying the condition:

For every $m \in I$ and every $m_1 > m$ in the Borel-order one has $m_1 \in I$.

Ideals whose gin is known

*) If $I = (f)$ and $\deg f = i$, then $\mathrm{gin}_\tau(I) = (x_1^i)$

*) If I is Borel-fixed then $\mathrm{gin}_\tau(I) = I$

*) In $\mathrm{char}\, 0$ and 2-variables the gin is the (lex-)segment. Hence $\mathrm{gin}(I)$ is determined by the HF of I.

*) If I is strongly stable non-segment then $\mathrm{gin}_\tau(I) = I$. So gin lex is not always the lex-segment.

*) If $I = g(J)$ for some $g \in \mathrm{GL}_n$ then $\mathrm{gin}_\tau(I) = \mathrm{gin}_\tau(J)$

*) If I is generated by r independent linear forms then $\mathrm{gin}_\tau(I) = (x_1, \ldots, x_r)$

*) If $I \subset R = K[x_1, \ldots, x_{n-1}]$ and $S = R[x_n]$ then $\mathrm{gin}_\tau(IS) = \mathrm{gin}_\tau(I)S$

Problem: Check with CoCoA.

In general:

(1) $\mathrm{gin}(I+J) \supseteq \mathrm{gin}(I) + \mathrm{gin}(J)$

(2) $\mathrm{gin}(IJ) \supseteq \mathrm{gin}(I)\,\mathrm{gin}(J)$

(3) $\mathrm{gin}(I \cap J) \subseteq \mathrm{gin}(I) \cap \mathrm{gin}(J)$

usually strict.

Problem: Check with CoCoA.

Problem: Given an ideal I shows that $\{\mathrm{gin}(I^i)\}_i \in \mathbb{N}$ is a filtration. Is the associated Rees ring Noetherian? Sometimes yes, but I do not know in general what is going on.

Problem: If $I \subset R = K[x_1, \ldots, x_n]$ and $J \subset S = K[y_1, \ldots, y_m]$ what is the relation between $\mathrm{gin}(I)$, $\mathrm{gin}(J)$ and $\mathrm{gin}(I+J)$? Say the term order is revlex. Any guess?