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Betti Numbers and Generic Initial Ideals

Lecture 2

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Automorphisms of polynomial rings

K infinite field.

$R = K[x_1, \ldots, x_n]$ is a K-algebra (i.e. a ring and a K-vector space)

A K-algebra homomorphism

$$g : R \longrightarrow R$$

is a ring homomorphism such that $g(a) = a$ for all $a \in K$.

g is determined by the images of x_i:
if $g(x_i) = g_i \in R$ then

$$g(F(x_1, \ldots, x_n)) = F(g_1, \ldots, g_n) \qquad (1)$$

Viceversa, for every $(g_1, \ldots, g_n) \in R^n$ the map (1) is a K-algebra homomorphism.

$$\{K - \text{alg.homomorphism } R \longrightarrow R\} \leftrightarrow R^n$$

$g \leftrightarrow (g_1, \ldots, g_n)$, $f \leftrightarrow (f_1, \ldots, f_n)$

composition of maps $f \circ g \leftrightarrow (h_1, \ldots, h_n)$

with $h_i = g_i(f_1, \ldots, f_n)$.

A K-algebra automorphism of R is a bijective K-algebra homomorphism

How do we decide whether $(g_1, \ldots, g_n)$ corresponds to a K-alg. automorphism?

It suffices to check that $(g_1, \ldots, g_n)$ induces a surjective map,
i.e. $K[g_1, \ldots, g_n] = K[x_1, \ldots, x_n]$
i.e. for every i there exists $f_i \in R$ such that $f_i(g_1, \ldots, g_n) = x_i$.

Jacobian Conjecture (char $K = 0$)

$g(x_i) = g_i$ is a K-alg automorphism

$\Leftrightarrow$

$\det(\partial g_i / \partial x_j) \in K \setminus \{0\}$.

$\Rightarrow$ is easy: if g is an automorphism, then there exists $f_1, \ldots, f_n$ so that $f_i(g_1, \ldots, g_n) = x_i$. Then use the chain rule: $J_f(g) J_g = I$ and hence $\det J_g \in K \setminus \{0\}$.

$\Leftarrow$ not known even for $n = 2$.

Problem: For a given g the Jacobian Conjecture can be checked using GB. Write a CoCoA program which does it. For algebra membership test, see [KR, Ch.3].

Graded Automorphisms of polynomial rings

A ring homomorphism $g : R \longrightarrow R$ is (standard) graded if it preserves homogeneity and degree.

K-algebra graded homomorphism

$$\updownarrow$$

$(g_1, \ldots, g_n)$ with g_i homogeneous of degree 1, $g_i = g_{1i}x_1 + \cdots + g_{ni}x_n$

$$\updownarrow$$

$$(g_{ij}) \in M_{nn}(K)$$

as semigroups (composition $=$ rows$\times$columns)

Conclusion:

K-algebra graded automorphisms of R

$$\updownarrow$$

$$\mathrm{GL}_n(K)$$

(as groups)

$g = (g_{ij}) \in \mathrm{GL}_n(K)$ acts on R by

$$g(x_i) = \sum_{j=1}^{n} g_{ji} x_j$$

$$g(F(x_1, \dots, x_n)) = F(g(x_1), \dots, g(x_n))$$

$R = K[x_1, \ldots, x_n]$

τ t.o. $x_1 > x_2 > \cdots > x_n$.

$\mathrm{in}_\tau = \mathrm{LT}_\tau$

I homogeneous ideal

$g \in \mathrm{GL}_n = \mathrm{GL}_n(K)$

$g(I)$ is an isomorphic copy of I

$g(I)$ is I written wrt a different system of coordinates.

$R/I \simeq R/g(I)$ as graded K-algebras.

Any graded K-algebra automorphism

$$R/I \longrightarrow R/I$$

is induced by a $g \in \mathrm{GL}_n$ such that $g(I) = I$.

Problem

$G \subseteq \mathrm{GL}_n$ a subgroup, we say that I is G-fixed if $g(I) = I$ for every $g \in G$.

Show that:

If G is generated (as a group) by a subset $W \subset G$, then I is G-fixed iff $g(I) = I$ for every $g \in W$.

I is G-fixed iff $g(I) \subset I$ for every $g \in G$.

$\bigcap_{g \in G} g(I)$ is the largest G-fixed ideal contained in I.

$\sum_{g \in G} g(I)$ is the smallest G-fixed ideal containing I.

Sums, products, intersections and colon of G-fixed ideals are G-fixed.

I is monomial iff I is T-fixed where $T = (K^*)^n$ is the diagonal subgroup (K is infinite).

If char $K = 0$, then the only GL_n-fixed ideals are the powers of $(x_1, \ldots, x_n)$.

For $G = S_n$ write a CoCoA function to check whether a given ideal is G-fixed.

Given an ideal I the set $G_I = \{g \in \mathrm{GL}_n : g(I) = I\}$ is a subgroup of G. Write a CoCoA function to compute G_I for a $I = (f)$ and f form of degree d in $K[x_1, x_2, x_3]$.

Generic initial ideals

Given τ and I consider the family

$$\mathrm{in}_\tau(g(I)) \quad g \in \mathrm{GL}_n$$

B_n=upper triangular invertible (Borel group)

U_n=upper triangular with 1 on the diagonal (unipotent group)

$U_n \subset B_n \subset \mathrm{GL}_n$

GL_n is Zariski-open in $\mathbb{A}^{n^2}$.

THM (Galligo, Bayer–Stillman)

1) For a generic $g \in \mathrm{GL}_n$ the ideal $\mathrm{in}_\tau(g(I))$ is "constant", i.e. there exists a non-empty Zariski-open subset U of GL_n such that

$$\mathrm{in}_\tau(g(I)) = \mathrm{in}_\tau(h(I)) \; \forall \; g, h \in U$$

Set $\mathrm{gin}_\tau(I) = \mathrm{in}_\tau(g(I))$ with $g \in U$

$\mathrm{gin}_\tau(I)$ is the generic initial ideal (gin) wrt τ.

$J = \mathrm{gin}_\tau(I)$.

2) J is Borel-fixed, i.e B_n-fixed.

3) $\mathrm{in}_\tau(g(I)) = J$ for a generic $g \in U_n$.

Proof 1): Take $G = (G_{ij})$ of size $n \times n$ where G_{ij} are variables over K. Consider $K' = K(G_{ij})$, $R' = K'[x_1, \ldots, x_n]$, $G \in \mathrm{GL}_n(K')$ and IR' the extension of I to R'.

Apply Buchberger algorithm to $G(IR')$,

The coefficients of the polynomials showing up are elements of K', i.e. rational functions in the variables G_{ij}.

F=the product of the numerators and denominators of the leading coefficients of the polynomials showing up in the GB computation.

If $g \in$ is in $\mathrm{GL}_n(K)$ and $F(g) \neq 0$ then the Buchberger algorithm applied to $g(I)$ reproduces the "same" GB, with G_{ij} replaced by g_{ij}. Hence $\mathrm{in}_\tau(g(I))$ is "equal " to $\mathrm{in}_\tau(G(I))$.

Set $U = \{g \in \mathrm{GL}_n : F(g) \neq 0\}$.

Example: $R = K[x,y]$, $I = (x^2, y^2)$, $\tau = x > y$.

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

$G(I)$ is generated by $(G_{11}x + G_{21}y)^2$ and $(G_{12}x + G_{22}y)^2$.

If $\operatorname{char} K \neq 2$ then $\operatorname{in}(G(I)) = (x^2, xy, y^3)$ and the "important " coefficients are $\det(G)$ and G_{11} and G_{12}.

If $\operatorname{char} K = 2$ then $G(I) = I$ and $\operatorname{in}(G(I)) = I$ the "important " coefficient is $\det G$. It follows:

$$\operatorname{gin}_\tau(I) = \begin{cases} (x^2, xy, y^3) & \operatorname{char} K \neq 2 \\ & g_{11}g_{12} \neq 0 \\ & \\ (x^2, y^2) & \operatorname{char} K = 2 \\ & \forall\ g \end{cases}$$

The proofs 2) and 3) require a better desription of $\text{gin}_\tau(I)$.

Given τ t.o. and V a K-subspace of R define

$$\text{in}_\tau(V) = \langle \text{in}_\tau(f) : f \in V \quad f \neq 0 \rangle$$

if V is an ideal then $\text{in}_\tau(V)$ is an ideal

if V is a K-algebra then $\text{in}_\tau(V)$ is a K-algebra.

If V has finite vector space dimension then $\dim V = \dim \text{in}_\tau(V)$

If I is an homogeneous ideal then $\text{in}_\tau(I) = \oplus_i \text{in}_\tau(I_i)$, equivalently,

$$\text{in}_\tau(I)_i = \text{in}_\tau(I_i)$$

Computing $\mathrm{in}_\tau(V)$ when V is finite dimensional=Gauss reduction.

Here is why: If V is generated by $f_1, \ldots, f_k$ then we apply Gauss reduction to $f_1, \ldots, f_k$ which consists of the iteration of the following step: whenever two polynomials f_i and f_j $(i < j)$ of the list have the same initial monomial then replace f_j with

$$f_j := f_j - (a/b) f_i$$

where a is the initial coefficient of f_j and b is that of f_i. Get rid of the 0's. At the end we get a system of generators (indeed a basis) of V say $f_1', \ldots, f_h'$, such that $\mathrm{in}_\tau(f_i') \neq \mathrm{in}(f_j')$ if $i \neq j$. Then

$$\mathrm{in}_\tau(V) = \langle \mathrm{in}_\tau(f_1'), \ldots, \mathrm{in}_\tau(f_h') \rangle$$

Example:

$n = 3$, $K = \mathbb{Q}$, $\tau = \mathrm{Lex}$, $V = \langle f_1, f_2, f_3 \rangle$ with

$$f_1 = x_1^2 + x_1x_2, \qquad f_2 = 2x_1^2 + x_2^2$$
$$f_3 = x_1^2 + 2x_1x_2 + x_1x_3$$

Gauss reduction

$$f_2 := f_2 - 2f_1 = -2x_1x_2 + x_2^2$$
$$f_3 := f_3 - f_1 = x_1x_2 + x_1x_3$$
$$f_3 := f_3 + 1/2 f_2 = x_1x_3 + 1/2x_2^2$$

$$f_1' = x_1^2 + x_1x_2, \qquad f_2' = -2x_1x_2 + x_2^2,$$
$$f_3' = x_1x_3 + 1/2x_2^2$$

$$\mathrm{in}_\tau(V) = \langle x_1^2, x_1x_2, x_1x_3 \rangle$$

Example: Same as before

$$f_1 = x_1^2 + x_1x_2, \qquad f_2 = 2x_1^2 + x_2^2$$
$$f_3 = x_1^2 + 2x_1x_2 + x_1x_3$$

M_V=matrix with 3 rows corresponding to f_1, f_2, f_3 and 4 columns corresponding to the monomials $x_1^2, x_1x_2, x_1x_3, x_2^2$ (Lex order).

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

then $[1,2,3|1,2,3]_{M_V} \neq 0$ and hence

$$\mathrm{in}_\tau(V) = \langle x_1^2, x_1x_2, x_1x_3 \rangle$$

V vector space of forms of degree i and dimension d. Wrt the monomial basis and given a basis of V, V is represented by a matrix $d \times \binom{n-1+i}{i}$, call it M_V. For distinct monomials $m_1, \dots, m_d$ of degree i we put

$[m_1, \dots, m_d]_V =$

$\det$ minor M_V with columns $m_1, \dots, m_d$

defined up to sign, "essentially" not depending on the given basis but only on V.

$$\mathrm{Supp}(V) =$$

$$\{\{m_1, \dots, m_d\} : [m_1, \dots, m_d]_V \neq 0\}$$

Alternative description: $\wedge^d V$ is a 1-dimensional subspace of $\wedge^d R_i$, and $\mathrm{Supp}(V)$ is the set of exterior monomials....

*1) Gauss reduction: Order colums according to τ. There exists $\{m_1, \dots, m_d\} \in \mathrm{Supp}(V)$ such that

$$m_1 > m_2 > \cdots > m_d$$

and $\forall$ $\{n_1, \dots, n_d\} \in \mathrm{Supp}(V)$ with

$$n_1 > n_2 > \cdots > n_d$$

one has

$$m_j \geq n_j \quad \forall \quad j$$

Call $\{m_1, \dots, m_d\}$ the $\max_\tau$ of $\mathrm{Supp}(V)$.

By construction,

$$\mathrm{in}_\tau(V) = \langle m_1, \dots, m_d \rangle$$

*2) “maximal and constant”

$$\mathrm{gin}_\tau(I_i) = \langle m_1, \dots, m_d \rangle$$

where $\{m_1, \dots, m_d\}$ is the $\max_\tau$ of $\mathrm{Supp}(G(I_i))$, G matrix of variables as in the proof of 1).

$\forall\ g \in \mathrm{GL}_n$ one has

$$\mathrm{Supp}(g(I_i)) \subseteq \mathrm{Supp}(G(I_i))$$

with $=$ for g generic.

$\forall\ g$ and $\forall\ \{n_1, \dots, n_d\} \in \mathrm{Supp}(g(I_i))$ with $n_1 > \cdots > n_d$ one has $n_j \leq m_j\ \forall\ j$

*3) Main Lemma: if V is a space of dimension d of forms of degree i and τ a t.o. then

$$\mathrm{Supp}(g(\mathrm{in}_\tau(V))) \subseteq \mathrm{Supp}(g(V))$$

for a generic $g \in \mathrm{GL}_n$.

Proof: For $G = (G_{ij})$ and give $\deg G_{ij} = e_j \in \mathbb{Z}^n$. For monomials $n_1, \ldots, n_d$, the element $[n_1, \ldots, n_d]_{G(\mathrm{in}_\tau(V))}$ is a multi-homogeneous polynomial in the G_{ij} and it is a multi-homogeneous component of $[n_1, \ldots, n_d]_{G(V)}$.

So if the first does not vanish, then the second does not vanish.

*4) For every space or ideal V and for every g lower tringular one has

$$\text{in}_\tau(g(V)) = \text{in}_\tau(V)$$

Proof: Let g be lower triangular. Then for every monomial m we have $\text{in}_\tau(g(m)) = m$. Hence for every polynomial f we have $\text{in}_\tau(g(f)) = \text{in}_\tau(f)$. Therefore for every vector space V $\text{in}_\tau(g(V)) = \text{in}_\tau(V)$.

*5) Given monomials $m_1 > \cdots > m_d$ set $V = \langle m_1, \ldots, m_d \rangle$. For g upper tringular and every $\{n_1 > \cdots > n_d\}$ in $\text{Supp}(g(V))$ one has $n_j \geq m_j$.

*6) $\text{Supp}(V) = \{\{m_1, \ldots, m_d\}\}$ iff $V = \langle m_1, \ldots, m_d \rangle$.

Proof 2): J is Borel-fixed if its homogeneous components are Borel-fixed. Consider J_i and write

$$J_i = \langle m_1, \ldots, m_d \rangle = \mathrm{in}_\tau(g_1(I_i))$$

with g_1 generic. Apply the Main Lemma to $V = g_1(I_i)$. We have:

$$\mathrm{Supp}(g(J_i)) \subseteq \mathrm{Supp}(gg_1(I_i))$$

with g generic.

By *2) we have that for all $g \in \mathrm{GL}_n$ every $\{n_1 > \cdots > n_d\} \in \mathrm{Supp}(gg_1(I_i)$ one has $n_i \leq m_i$.

(*) Hence for all $g \in \mathrm{GL}_n$ and every $\{n_1 > \cdots > n_d\} \in \mathrm{Supp}(g(J_i))$ one has $n_i \leq m_i$.

We may take $g = h \in B_n$. Combining (*) and *5) we have that $\mathrm{Supp}(h(J_i)) = \{\{m_1, \ldots, m_d\}\}$ for all $h \in B_n$. By 6*) we conclude that $h(J_i) = J_i$ for all $h \in B_n$.

Proof 3: Let $g \in \mathrm{GL}_n$ generic so that $\mathrm{in}_\tau(g(I)) = J$ and so that g has a LDU-decomposition. Say $g = ab$ with a lower (with arbitrary diagonal elements) and $b \in U_n$. Then by *4)

$$J = \mathrm{in}_\tau(g(I)) = \mathrm{in}_\tau(ab(I)) = \mathrm{in}_\tau(b(I))$$

So J is obtained with a $b \in U_n$. By the "maximality" of J, it is obtained also by the generic $b \in U_n$.

How to compute the gin?

1) $g \in \mathrm{GL}_n$ "random" upper triangular and compute $\mathrm{in}_\tau(g(I))$;

   the result is the $\mathrm{gin}_\tau(I)$ only with high probability

2) $G = (G_{ij})$ upper triangular with G_{ij}, $i < j$, algebraically independent over K and compute $\mathrm{in}_\tau(G(I))$

   – too many variables –

Problem: Find an efficent algorithm to compute gin

Example: Compute gin lex and revlex of $I = (x^2, y^2, z^2)$. Take an upper triangular of variables:

$$G = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

$$G(I) = (x^2, (ax + y)^2, (bx + cy + z)^2)$$

We have to determine

$$\mathrm{in}(G(I))$$

From degree 4 on I contains everything, so the action takes place in degree 2 and 3. Computing the supports of $G(I)_2$ and $G(I)_3$ one gets the following results:

$$\mathrm{gin}_{\mathrm{lex}}(I) = \begin{cases} (x^2, xy, xz, y^3, y^2z, yz^2, z^4) \\ \mathrm{char}\, K \neq 2,3 \quad abc(ac-b) \neq 0 \\ \\ (x^2, xy, xz, y^3, y^2z, z^3) \\ \mathrm{char}\, K = 3 \;\; ab(ac+b) \neq 0 \\ \\ I \\ \mathrm{char}\, K = 2 \end{cases}$$

$$\mathrm{gin}_{\mathrm{rlex}}(I) = \begin{cases} (x^2, xy, y^2, xz^2, yz^2, z^4) \\ \mathrm{char}\, K \neq 2,3 \;\; c(ac-b) \neq 0 \\ \\ (x^2, xy, y^2, xz^2, z^3) \\ \mathrm{char}\, K = 3, c(ac-b)(ac+b) \neq 0 \\ \\ I \\ \mathrm{char}\, K = 2 \end{cases}$$

In this example we see that:

*) The gin depends on the term order

*) The gin revlex is “smaller” than the gin lex (true in general)

*) In $\mathrm{char}\, 0$ the gins are “segments” (not true in general)