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Betti Numbers and Generic Initial Ideals

Lecture 1

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Generic (?)

Generic polynomials? Generic matrices?

K infinite field
$R = K[x_1, \ldots, x_n]$ polynomial ring
$\mathbb{A}^n =$ affine space over K, i.e.
as a set K^n + Zariski topology

Remark $F \in R$, $F \neq 0$ then there exists $(a_1, \ldots, a_n) \in \mathbb{A}^n$ such that $F(a_1, \ldots, a_n) \neq 0$.

Roughly, given F, a point of $\mathbb{A}^n$ taken at "random" is not a solution of the polynomial equation $F = 0$.

On the other hand, if K is finite $K = \{a_1, \ldots, a_q\}$ and

$$F = \Pi_j (x_1 - a_j)$$

then $F \neq 0$ as a polynomial and vanishes as a function on $\mathbb{A}^n$.

Assume:

$Y = \mathbb{A}^n$ or an irreducible subvariety of $\mathbb{A}^n$, or a non-empty Zariski-open subset of an irreducible variety of $\mathbb{A}^n$.

A family $\{F_y\}$ of objects (e.g. polynomials, matrices, maps, etc..) parameterized by $y \in Y$.

P a property (e.g. to be irreducible, to have a certain rank, to be injective, etc...)

Definition We say that the generic F_y has property P provided:

there exists a non-empty Zariski-open subset U of Y such that F_y has property P for all $y \in U$.

Equivalently, there exists a polynomial G not identically zero on Y such that

$y \in Y$ with $G(y) \neq 0 \Rightarrow F_y$ has P

Intuitively, the generic F_y has proprerty P if F_y has property P for "almost all" $y \in Y$.

The set of the $y \in Y$ such that F_y does not have property P is "very small": it is contained is a Zariski-closed proper subset of Y.

Warning: Saying that the generic F_y has property P does not mean that the set $\mathbf{W}$ of y such that F_y has property P is non-empty Zariski-open. It means $\mathbf{W}$ contains a non-empty Zariski-open.

Y is irreducible $\Rightarrow$ non-empty Zariski-open are dense.

The intersection of finitely many non-empty Zariski-open subsets of Y is non-empty Zariski-open.

So if the generic F_y has property P and the generic F_y has property Q then the generic F_y has property P and Q.

Example (over $\mathbb{C}$)

*) The generic polynomial (in one variable) of degree 2 is not a square.

$Y = \{(a,b,c) \in \mathbb{A}^3 : a \neq 0\}$, $y = (a,b,c) \in Y$ define

$$F_y = ax^2 + bx + c$$

F_y is not a square iff $b^2 - 4ac \neq 0$.

*) The generic polynomial of degree 2 does not vanish at 1.

$$a + b + c \neq 0$$

*) The generic polynomial of degree 2 is not a square and does not vanish at 1

$$(a + b + c)(b^2 - 4ac) \neq 0$$

Example (different parameterization) The generic polynomial of degree 2 is not a square and does not vanish at 1.

$Y \subset \mathbb{A}^3$, $y = (a, b, c) \in Y$ with $a \neq 0$ define

$$F_y = a(x - b)(x - c)$$

F_y is not a square and does not vanish at 1 if $(b - 1)(c - 1)(b - c) \neq 0$

This parameterization is $2:1$ "generically".

$y = (a, b, c)$ and $z = (a, c, b)$ then $F_y = F_z$.

Example For a polynomial f with complex coefficients consider the property: $P : f(1) \notin \mathbb{R}$

Does the generic polynomial f of degree 2 (and complex coefficients) satisfy property P?

$f = ax^2+bx+c$ parametrized by with $(a, b, c) \in \mathbb{A}^3_{\mathbb{C}}$ and $a \neq 0$

The condition is $a + b + c \notin \mathbb{R}$.

It turns out (provide details) that the locus where the property holds does not contain a non-empty Zariski-open.

In this example: the generic polynomial does not have P and does not have “not P”. There is no generic behaviour in the given sense.

Things change is we parameterize with $\mathbb{A}^6_{\mathbb{R}}$.

Example For $y \in Y = \mathbb{A}^n$ let

$$f_y : \mathbb{C}^s \longrightarrow \mathbb{C}^r$$

be a linear map described by a matrix $Z_y = (z_{ij}(y))$ of size $r \times s$ whose entries are polynomials in the coordinates of y.

Then the generic f_y has "constant and maximal" rank, that is,

$$c = \max\{\mathrm{rk}\, f_y : y \in Y\}$$

then the generic f_y has rank c.

Consider variables $(x_1, \ldots, x_n)$ and look at the rank of $Z_x = (z_{ij}(x))$ as a matrix of the ring $K[x_1, \ldots, x_n]$.

Say the rank is v. It means that all the $v+1$-minors of Z_x are 0 and that there exists a non-zero minor $M(x)$ of size v.

For all $y \in \mathbb{A}^n$ we have the rank of Z_y is $\leq v$.

If $y \in \mathbb{A}^s$ and $M(y) \neq 0$ then rank of Z_y is v.

U is the subset of Y defined by $M(x) \neq 0$.

For instance, if $n = 3$ and $y = (y_1, y_2, y_3) \in \mathbb{A}^3$ and

$$f_y : \mathbb{C}^4 \longrightarrow \mathbb{C}^4$$

defined by

$$f_y(e_1) = y_3 e_1 + y_3 e_4$$

$$f_y(e_1) = y_3 e_1 + y_3 e_2 + y_1 e_4$$

$$f_y(e_3) = y_2 e_1 + y_2 e_4$$

$$f_y(e_3) = y_1 e_3 + y_1 e_4$$

then

$$Z_y = \begin{pmatrix} y_3 & y_3 & y_2 & 0 \\ 0 & y_3 & 0 & 0 \\ 0 & 0 & 0 & y_1 \\ y_3 & y_1 & y_2 & y_1 \end{pmatrix}$$

and

$$Z_x = \begin{pmatrix} x_3 & x_3 & x_2 & 0 \\ 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & x_1 \\ x_3 & x_1 & x_2 & x_1 \end{pmatrix}$$

where x_1, x_2, x_3 are variables.

$$\det Z_x = 0$$

$x_3^2 x_1$ is a 3-minor of Z_x

So the rank of Z_x is 3 and the we can conclude that the generic f_y has rank 3.

Warning: this does not mean that the image of the generic f_y is independent of y.

Problem

Let $F = \mathrm{Hom}(\mathbb{C}^s, \mathbb{C}^r)$ parameterized by $\mathbb{A}^{rs}$.

Set $P =$" $(1, 0, \ldots, 0) \in \mathrm{Image}\, f$".

Does the generic $f \in F$ have the property P?

Set $P_1 =$ " $(1, 0, \ldots, 0) \notin \mathrm{Image}\, f$".

Does the generic $f \in F$ have the property P_1?

forms=homogeneous polynomials
quadrics=forms of degree 2
cubics=forms of degree 3

Example The dimension of the space of cubics belonging to the ideal of 4 generic quadrics in 3 variables is "constant and maximal".

a quadric in 3 variables x_1, x_2, x_3 needs 6 coefficients

$$a_1x_1^2 + a_2x_1x_2 + \cdots + a_6x_3^2$$

4 quadrics need 24 coefficients

Each $p \in \mathbb{A}^{24}$ corresponds to a set $q = q_1, \ldots, q_4$ of 4 quadrics.

We look at the dimension of the vector space generated by the 12 cubics x_iq_j $i = 1, 2, 3$, $j = 1, 2, 3, 4$.

Consider the matrix X_p whose columns corresponds to monomials of degree 3 and rows to the $x_i q_j$ and in the position corresponding to the monomial m and to $x_i q_j$ we put the coefficient of m in $x_i q_j$.

Note that X_p is a 12×10 matrix whose coefficients are polynomial functions on the 24 coefficients,the coordinates of p.

The dimension of the space of cubics belonging to the ideal generated by $q_1, \ldots, q_4$ is the rank of X_p.

As in the previous example, we have that the rank is maximal for a generic p.

Problem In the previous example is the generic rank 10? To prove it, it is enough to provide an example of an ideal generated of 4 quadrics and containing all the cubics.

Problem Show that the ideal generated by 5 generic quadrics in 4 variables does not have a GB of quadrics no matter what the term order and the coordinate system is.

Problem Show that the ideal generated by 7 generic quadrics in 4 variables has a GB of quadrics (with respect to a properly chosen term order and coordinate system).

Problem Show that the ideal generated by 3 generic quadrics in 3 variables does not contain the square of a linear form.

Problem:

$R = K[x_1, \ldots, x_n]$

R_i space of forms of degree i.

Given $V \subseteq R_t$ with $\dim V = d$ what is the dimension of VR_j?

Obvious upper bound:

$$\dim VR_j \leq \min(\dim R_{t+j}, d \dim R_j)$$

Expected:

$$\dim VR_j = \min(\dim R_{t+j}, d \dim R_j)$$

for generic V and $j < t$. (not known for $n > 3$ and $j > 1$).

This is a special case of the Fröberg conjecture.

Problem:

$R = K[x_1, \ldots, x_n]$

R_i space of forms of degree i.

Given $V \subseteq R_t$ with $\dim V = d$ what is the dimension of V^2?

Obvious upper bound:

$$\dim V^2 \leq \min(\dim R_{2t}, d(d+1)/2)$$

Expected(?) $=$ for generic V. False in general.

$n = 4, t = 2$ $d = 8$: V^2 has 36 generators and lives in R_4 which has dimension 35. We expect $V^2 = R_4$ but instead $\dim V^2 = 34$ for generic V. Check (or prove) this. Why does this happen? This is the only exception known (to me). It has been discovered by L.Chiantini.

Problem:

$V = \mathbb{C}^n$

$E = \oplus_{i=0}^n E_i$ exterior algebra.

$E = K[e_1, \dots, e_n]$ with $e_i e_j = -e_j e_i$.

Consider $\varphi_f : E_d \longrightarrow E_{2d}$ the multiplication map $f \in E_d$ with d odd. Observe that $f^2 = 0$. What is the rank of φ_f?

We have upper bound:

$$\operatorname{rk} \varphi_f \leq \min\{\binom{n}{d} - 1, \binom{n}{2d}\}$$

What do you expect for generic f? Check the case $d = 3, n = 9$. I learnt this problem from Winfried Bruns.

Example A generic square matrix has a LDU-decomposition and a UDL-decomposition

LDU=Lower tr.*Diagonal*Upper tr.

UDL=Upper tr.*Diagonal*lower tr.

(triangular matrices have 1 on the diagonal).

$n \times n$ matrices $\leftrightarrow \mathbb{A}^{n^2}$.

Denote by $[a_1, \ldots, a_s | b_1, \ldots, b_s]_X$ the minor of X with rows $a_1, \ldots, a_s$ and columns $b_1, \ldots, b_s$.

Precise statement:
For every X $n \times n$ matrix such that

$$[1, 2, \ldots, i | 1, 2, \ldots, i]_X \neq 0$$

for all $i = 1, \ldots, n-1$ then there exist L, D, U such that

$$X = LDU$$

L, D, U are uniquely determined by X and given by explicit formuals. For instance,

$$D_{ii} = \frac{[1,\ldots,i|1,\ldots,i]_X}{[1,\ldots,i-1|1,\ldots,i-1]_X}$$

Similarly for UDL-decomposition, the conditions being

$$[i, i+1, \ldots, n|i, i+1, \ldots, n]_X \neq 0$$

for all $i = 2, \ldots, n$

Note: For $n = 2$

X has a LDU-dec. iff $(x_{11} \neq 0)$ or $(x_{11} = x_{12} = x_{21} = 0)$

The set

$$\{X : X \text{ has a LDU-decomposition}\}$$

is not Zariski-open, it contains the non-empty Zariski-open given by $x_{11} \neq 0$

Problem Check that the following instructions compute the LDU decomposition in the generic case.

```
Define DSubMat(X,A,B);
Return
Det(Mat[[X[I,J] | J In B] | I In A]);
EndDefine;

N:=6;
Use Q[x[1..N,1..N]];

X:=Mat[[x[I,J] |J In 1..N] | I In 1..N];

--define D
D:=Identity(N);
PM:=[DSubMat(X,1..K,1..K) | K In 1..N];
D[1,1]:=PM[1];
For I:=2 To N Do
D[I,I]:=PM[I]/PM[I-1];
EndFor;
```

```
--define L
L:=Identity(N);
For I:=2 To N Do
For J:=1 To I-1 Do
L[I,J]:=
DSubMat(X,Concat(1..(J-1),[I]),1..J)
/PM[J];
EndFor; EndFor;

--define U
U:=Identity(N);
For I:=2 To N Do
For J:=1 To I-1 Do
U[J,I]:=
DSubMat(X,1..J,Concat(1..(J-1),[I]))
/PM[J];
EndFor; EndFor;

L*D*U=X;
```