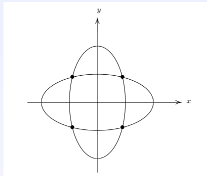
Two conics I



Example

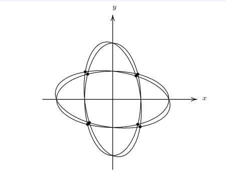
Consider the polynomial system

$$\begin{array}{rcccl} f_1 & = & \frac{1}{4}x^2 + y^2 - 1 & = & 0 \\ f_2 & = & x^2 + \frac{1}{4}y^2 - 1 & = & 0 \end{array}$$

$\mathbb{X} = \mathcal{Z}(f_1) \cap \mathcal{Z}(f_2)$ consists of the four points $\mathbb{X} = \{(\pm\sqrt{4/5}, \pm\sqrt{4/5})\}$.

The set $\{x^2 - \frac{4}{5}, y^2 - \frac{4}{5}\}$ is the reduced Gröbner basis of the ideal $I = (f_1, f_2) \subseteq \mathbb{C}[x, y]$ with respect to $\sigma = \mathtt{DegRevLex}$. Therefore we have $\mathrm{LT}_\sigma(I) = (x^2, y^2)$, and the residue classes of the terms in $\mathbb{T}^2 \setminus \mathrm{LT}_\sigma\{I\} = \{1, x, y, xy\}$ form a $\mathbb{C}$ -vector space basis of $\mathbb{C}[x, y]/I$.

Two conics II



Now consider the slightly perturbed polynomial system

$$\begin{aligned} \tilde{f}_1 &= \tfrac{1}{4}x^2 + y^2 + \varepsilon\, xy - 1 &= 0 \\ \tilde{f}_2 &= x^2 + \tfrac{1}{4}y^2 + \varepsilon\, xy - 1 &= 0 \end{aligned}$$

where ε is a small number.

The intersection of $\mathcal{Z}(\tilde{f}_1)$ and $\mathcal{Z}(\tilde{f}_2)$ consists of four perturbed points $\widetilde{\mathbb{X}}$ close to those in $\mathbb{X}$.

Two conics III

- This time the ideal $\tilde{I} = (\tilde{f}_1, \tilde{f}_2)$ has the reduced σ -Gröbner basis

$$\{x^2 - y^2,\ xy + \tfrac{5}{4\varepsilon}\, y^2 - \tfrac{1}{\varepsilon},\ y^3 - \tfrac{16\varepsilon}{16\varepsilon^2-25}\, x + \tfrac{20}{16\varepsilon^2-25}\, y\}$$

- Moreover, we have $\mathrm{LT}_\sigma(\tilde{I}) = (x^2,\ xy,\ y^3)$ and $\mathbb{T}^2 \setminus \mathrm{LT}_\sigma\{\tilde{I}\} = \{1,\ x,\ y,\ y^2\}$.
- A small change in the coefficients of f_1 and f_2 has led to a big change in the Gröbner basis of (f_1, f_2) and in the associated vector space basis of $\mathbb{C}[x, y]/(f_1, f_2)$, although the zeros of the system have not changed much.
- Numerical analysts call this kind of unstable behaviour a representation singularity.

BB: Introduction I

- The basic idea of border basis theory is to describe a zero-dimensional ring P/I by an order ideal of monomials $\mathcal{O}$ whose residue classes form a K-basis of P/I and by the multiplication table of this basis.
- We can build border basis theory in the following way. Let $\mathcal{O} = \{t_1, \dots, t_\mu\}$ be an order ideal and $\partial\mathcal{O} = \{b_1, \dots, b_\nu\}$ its border. A border prebasis consists of polynomials of the form $g_j = b_j - \sum_{i=1}^{\mu} \alpha_{ij} t_i$ with $\alpha_{ij} \in K$ and can be used for border division.
- It is a border basis if and only if the residue classes of $\{t_1, \dots, t_\mu\}$ form a K-basis of P/I.
- If $\mathcal{O}$ represents a vector space basis of P/I, then an $\mathcal{O}$-border basis of I exists, is uniquely determined, and generates the ideal.
- Furthermore, it is then possible to define and compute normal forms with respect to $\mathcal{O}$.

BB: Introduction II

Let K be a field, let $P = K[x_1, \dots, x_n]$ be standard graded, and let $\mathbb{T}^n$ be the monoid of terms in P.

Definition

Let $\mathcal{O}$ be a non-empty subset of $\mathbb{T}^n$.

1. The closure of $\mathcal{O}$ is the set $\overline{\mathcal{O}}$ of all terms in $\mathbb{T}^n$ which divide one of the terms of $\mathcal{O}$.
2. The set $\mathcal{O}$ is called an order ideal if $\overline{\mathcal{O}} = \mathcal{O}$, i.e. $\mathcal{O}$ is closed under forming divisors.

Definition

A set of polynomials $G = \{g_1, \dots, g_\nu\}$ in P is called an $\mathcal{O}$-border prebasis if the polynomials have the form $g_j = b_j - \sum_{i=1}^{\mu} \alpha_{ij} t_i$ with $\alpha_{ij} \in K$ for $1 \le i \le \mu$ and $1 \le j \le \nu$.

BB: Introduction III

Definition

Let $G = \{g_1, \dots, g_\nu\}$ be an $\mathcal{O}$-border prebasis, let $\mathcal{G}$ be the tuple $(g_1, \dots, g_\nu)$, and let $I \subseteq P$ be an ideal containing G. The set G or the tuple $\mathcal{G}$ is called an $\mathcal{O}$-border basis of I if one of the following equivalent conditions is satisfied.

1. The residue classes $\overline{\mathcal{O}} = \{\bar{t}_1, \dots, \bar{t}_\mu\}$ form a K-vector space basis of P/I.
2. We have $I \cap \langle \mathcal{O} \rangle_K = \{0\}$.
3. We have $P = I \oplus \langle \mathcal{O} \rangle_K$.

Existence and uniqueness

Proposition (Existence and Uniqueness of Border Bases)

Let $\mathcal{O} = \{t_1, \dots, t_\mu\}$ be an order ideal, let $I \subseteq P$ be a zero-dimensional ideal, and assume that the residue classes of the elements of $\mathcal{O}$ form a K-vector space basis of P/I.

1. *There exists a unique $\mathcal{O}$-border basis of I.*
2. *Let G be an $\mathcal{O}$-border prebasis whose elements are in I. Then G is the $\mathcal{O}$-border basis of I.*
3. *Let k be the field of definition of I. Then the $\mathcal{O}$-border basis of I is contained in $k[x_1, \dots, x_n]$.*

Proposition

Let σ be a term ordering on $\mathbb{T}^n$, and let $\mathcal{O}_\sigma(I)$ be the order ideal $\mathbb{T}^n \setminus \mathrm{LT}_\sigma\{I\}$. Then there exists a unique $\mathcal{O}_\sigma(I)$-border basis G of I, and the reduced σ-Gröbner basis of I is the subset of G corresponding to the corners of $\mathcal{O}_\sigma(I)$.

Existence and uniqueness

Border bases can be characterized imitating Gröbner bases theory. So we can use special generation, rewrite relations, syzygies and an important new feature.

Definition

Let $G = \{g_1, \ldots, g_\nu\}$ be an $\mathcal{O}$-border prebasis, i.e. let $g_j = b_j - \sum_{i=1}^{\mu} \alpha_{ij} t_i$ with $\alpha_{ij} \in K$ for $i = 1, \ldots, \mu$ and $j = 1, \ldots, \nu$. Given $r \in \{1, \ldots, n\}$, we define the r^{th} **formal multiplication matrix** $\mathcal{X}_r = (\xi_{k\ell}^{(r)})$ of G by

$$\xi_{k\ell}^{(r)} = \begin{cases} \delta_{ki}, & \text{if} \quad x_r\, t_\ell = t_i \\ \alpha_{kj}, & \text{if} \quad x_r\, t_\ell = b_j \end{cases}$$

Here we let $\delta_{ki} = 1$ if $k = i$ and $\delta_{ki} = 0$ otherwise.

Commuting matrices

The following is a fundamental fact.

Theorem (Border Bases and Commuting Matrices)

Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ *be an order ideal, let* $G = \{g_1, \ldots, g_\nu\}$ *be an* $\mathcal{O}$ *-border prebasis, and let* $I = (g_1, \ldots, g_\nu)$ *. Then the following conditions are equivalent.*

1. *The set* G *is an* $\mathcal{O}$ *-border basis of* I *.*
2. *The formal multiplication matrices of* G *are pairwise commuting.*

In that case the formal multiplication matrices represent the multiplication endomorphisms of P/I *with respect to the basis* $\{\bar{t}_1, \ldots, \bar{t}_\mu\}$ *.*

BB Algorithm

The most important feature of border bases is, of course, that they can also be computed.

Theorem (The Border Basis Algorithm)

Let $I \subseteq P$ be a zero-dimensional ideal generated by a set of non-zero polynomials $\{f_1, \ldots, f_s\}$, and let σ be a degree compatible term ordering. Consider the following sequence of instructions.

1. *Let $V_0 \subseteq P$ be the K-vector subspace generated by $\{f_1, \ldots, f_s\}$.*
2. *Let $d = \max\{\deg(t) \mid t \in \mathrm{Supp}(f_1) \cup \cdots \cup \mathrm{Supp}(f_s)\}$ and $\mathcal{L} = \mathbb{T}^n_{\leq d}$.*
3. *For $i = 0, 1, 2, \ldots$ compute $V_{i+1} = (V_i + x_1 V_i + \cdots + x_n V_i) \cap \langle \mathcal{L} \rangle_K$ until $V_{i+1} = V_i$.*
4. *Compute an order ideal $\mathcal{O} \subseteq \mathcal{L}$ such that the residue classes of the terms in $\mathcal{O}$ form a K-vector space basis of $\langle \mathcal{L} \rangle_K / V_i$.*
5. *Check whether $\partial\mathcal{O} \subseteq \mathcal{L}$. If this is not the case, increase d by one, replace $\mathcal{L}$ by $\mathbb{T}^n_{\leq d}$, replace V_0 by V_i, and continue with step 3).*
6. *Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ and $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$. For $j = 1, \ldots, \nu$, use linear algebra to compute the representation $\bar{b}_j = \sum_{i=1}^{\mu} \alpha_{ij} \bar{t}_i$ of $\bar{b}_j \in \langle \mathcal{L} \rangle_K / V_i$ in terms of the basis $\{\bar{t}_1, \ldots, \bar{t}_\mu\}$ and let $g_j = b_j - \sum_{i=1}^{\mu} \alpha_{ij} t_i$. Then let $G = \{g_1, \ldots, g_\nu\}$, return the pair $(\mathcal{O}, G)$, and stop.*

This is an algorithm which returns a pair $(\mathcal{O}, G)$ where $\mathcal{O}$ is an order ideal and G is the $\mathcal{O}$-border basis of I.

BB Variety

The following material is taken from [KR3]. Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal in $\mathbb{T}^n$, and let $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ be its border. We define a moduli space for *all* zero-dimensional ideals having an $\mathcal{O}$-border basis.

Definition

Let $\{c_{ij} \mid 1 \leq i \leq \mu,\ 1 \leq j \leq \nu\}$ be a set of further indeterminates.

1. The generic $\mathcal{O}$-border prebasis is the set of polynomials $G = \{g_1, \ldots, g_\nu\}$ in $Q = K[x_1, \ldots, x_n, c_{11}, \ldots, c_{\mu\nu}]$ given by

$$g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$$

2. For $k = 1, \ldots, n$, let $\mathcal{A}_k \in \mathrm{Mat}_\mu(K[c_{ij}])$ be the k^{th} formal multiplication matrix associated to G.
3. The affine variety $\mathbb{B}_\mathcal{O} \subseteq K^{\mu\nu}$ defined by the ideal $I(\mathbb{B}_\mathcal{O})$ generated by the entries of the matrices $\mathcal{A}_k\mathcal{A}_\ell - \mathcal{A}_\ell\mathcal{A}_k$ with $1 \leq k < \ell \leq n$ is called the $\mathcal{O}$-border basis variety.

Universal family

Definition

Let $G = \{g_1, \dots, g_\nu\} \subset Q = K[x_1, \dots, x_n, c_{11}, \dots, c_{\mu\nu}]$ with $g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$ for $j = 1, \dots, \nu$ be the generic $\mathcal{O}$-border prebasis. Then the natural homomorphism of K-algebras

$$\Phi : K[c_{11}, \dots, c_{\mu\nu}]/I(\mathbb{B}_{\mathcal{O}}) \longrightarrow K[x_1, \dots, x_n, c_{11}, \dots, c_{\mu\nu}]/(I(\mathbb{B}_{\mathcal{O}}) + \langle g_1, \dots, g_\nu \rangle)$$

is called the universal $\mathcal{O}$-border basis family.

The fibers of the basis family are the quotient rings P/I for which I is a zero-dimensional ideal which has an $\mathcal{O}$-border basis.

Theorem

Let $B = K[c_{11}, \dots, c_{\mu\nu}]/I(\mathbb{B}_{\mathcal{O}})$ be the affine coordinate ring of the $\mathcal{O}$-border basis variety and $\Phi : B \longrightarrow B[x_1, \dots, x_n]/\langle g_1, \dots, g_\nu \rangle$ the universal $\mathcal{O}$-border basis family. Then the residue classes of the elements of $\mathcal{O}$ are a B-module basis of $B[x_1, \dots, x_n]/\langle g_1, \dots, g_\nu \rangle$. In particular, the map Φ is a flat homomorphism.

Philosophy

- In the paper [KR3] we study properties of the border basis variety. For instance we show that in some cases it is an affine space.
- Suppose we are computing the border basis of an ideal of points I in P. We have a basis $\overline{\mathcal{O}}$ of the quotient ring. If we move the points slightly, $\overline{\mathcal{O}}$ is still a basis of the *perturbed ideal* $\tilde{I}$, since the evaluation matrix of the elements of $\mathcal{O}$ at the points has determinant different from zero.
- Consequently, the border basis is modified slightly into a new border basis of a zero-dimensional ideal. This sentence can be rephrased by saying that moving the points moves the border basis, and the movement traces a path inside the border variety.
- On the other hand, if we perturb the equations of the border basis, we are not at all guaranteed that we keep inside the border variety. What we can say is that the multiplication matrices almost commute, but most likely the new ideal is the unit ideal. This is an important remark since rarely the border variety is an affine space.
- One philosophical question here is the following. When we compute with approximate data, do we want to move along the border variety, or are we content to simply stay close to the border variety?

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