History I

- In the 1970s and early 1980s, algebraic geometry was essentially synonymous with Grothendieck's scheme theory.
- Most geometers were busily developing new cohomology theories and proving vanishing theorems in them; some had never seen the equations of a non-trivial example for their theories.
- In this atmosphere the few hardy folks who dared to give talks about such down-to-earth topics as finite sets of points in affine or projective spaces were confronted with disinterest or even ridicule.
- Now, about a quarter of a century later, the tide has turned completely: finite sets of points are an active and well-respected branch of algebraic geometry. How did this change come about?

History II

- When one starts to reduce deep problems in algebraic geometry to their essential parts, it frequently turns out that at their core lies a question which has been studied for a long time, and sometimes this question is related to finite sets of points.
- For instance, already in the eighteenth century G. Cramer and L. Euler discussed (in their correspondence) a phenomenon which is nowadays called the Cayley-Bacharach property.
- Later, in 1843, A. Cayley formulated a theorem which was a vast generalization of what Cramer and Euler had stumbled upon. But unfortunately the claim was false and the proof invalid.
- This was not corrected until 1886, when I. Bacharach used M. Noether's " $A\Phi + B\Psi$ "-theorem to give a correct statement and a true proof.
- Many decades later it turned out that the Cayley-Bacharach property is connected to the "Gorenstein property" of the homogeneous coordinate ring of a projective point set, and today a generalization of this property is central to an important conjecture by D. Eisenbud, M. Green and J. Harris.

History III

- What does all of this have to do with Computational Commutative Algebra?
- We think that finite sets of points provide excellent examples for the ways in which computer algebra methods can be applied.
- They show up in many branches of mathematics besides algebraic geometry, for instance in interpolation, coding theory, and statistics.
- Efficient algorithms help us to compute with larger and larger point sets, to check results and conjectures, and to discover new ones.
- And today we have a new entry in the game: ideals of finite sets of points with approximate coordinates.

Separators and Interpolators

We start with a finite set of points $\mathbb{X}$ in the affine space and call it an affine point set. Its vanishing ideal in P is called $\mathcal{I}(\mathbb{X})$.

If we want to perform polynomial interpolation, we also need to know the following polynomials.

Definition

Let $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq K^n$ be an affine point set, and let $\mathcal{X}$ be the tuple $(p_1, \dots, p_s)$.

1. Let $i \in \{1, \dots, s\}$. A polynomial $f \in P$ is called a separator of p_i from $\mathbb{X} \setminus p_i$ if $f(p_i) = 1$ and $f(p_j) = 0$ for $j \neq i$.
2. Let $a_1, \dots, a_s \in K$. A polynomial $f \in P$ is called an interpolator for the tuple $(a_1, \dots, a_s)$ at $\mathcal{X}$ if $f(p_i) = a_i$ for $i = 1, \dots, s$.

Separators exist

It is clear that separators and interpolators are not unique. Two separators of p_i and two interpolators for a tuple $(a_1, \ldots, a_s) \in K^s$ differ by an element of $\mathcal{I}(\mathbb{X})$.

Proposition

Let $\mathbb{X} = \{p_1, \ldots, p_s\} \subseteq K^n$ *be an affine point set, and let* $\mathcal{X}$ *be the tuple* $(p_1, \ldots, p_s)$.

1. *For every* $i \in \{1, \ldots, s\}$ *, there exists a separator of* p_i *from* $\mathbb{X} \setminus p_i$.
2. *For every* $(a_1, \ldots, a_s) \in K^s$ *, there exists an interpolator for* $(a_1, \ldots, a_s)$ *at* $\mathcal{X}$.
3. *Let* $\mathbb{Y}$ *be an affine point set contained in* $\mathbb{X}$ *. For every* $p_i \in \mathbb{X}$ *, let* $f_i \in P$ *be a separator of* p_i *from* $\mathbb{X} \setminus p_i$ *, and let* $f_{\mathbb{X}\setminus\mathbb{Y}} = \sum_{p_i \in \mathbb{X}\setminus\mathbb{Y}} f_i$ *. Then we have* $\mathcal{I}(\mathbb{Y}) = \mathcal{I}(\mathbb{X}) + (f_{\mathbb{X}\setminus\mathbb{Y}})$.

The BM Algorithm I

Theorem (The Buchberger-Möller Algorithm)

Let σ be a term ordering on $\mathbb{T}^n$, and let $\mathbb{X} = \{p_1, \ldots, p_s\}$ be an affine point set in K^n whose points $p_i = (c_{i1}, \ldots, c_{in})$ are given via their coordinates $c_{ij} \in K$. Consider the following sequence of instructions.

(1) *Let $\mathcal{G} = \emptyset$, $O = \emptyset$, $\mathcal{S} = \emptyset$, $L = \{1\}$, and let $\mathcal{M} = (m_{ij}) \in \mathrm{Mat}_{0,s}(K)$ be a matrix having s columns and initially zero rows.*

(2) *If $L = \emptyset$, return the pair $(\mathcal{G}, O)$ and stop. Otherwise, choose the term $t = \min_\sigma(L)$ and delete it from L.*

(3) *Compute the evaluation vector $(t(p_1), \ldots, t(p_s)) \in K^s$ and reduce it against the rows of $\mathcal{M}$ to obtain*

$$(v_1, \ldots, v_s) = (t(p_1), \ldots, t(p_s)) - \sum_i a_i\, (m_{i1}, \ldots, m_{is})$$

with $a_i \in K$.

(4) *If $(v_1, \ldots, v_s) = (0, \ldots, 0)$ then append the polynomial $t - \sum_i a_i s_i$ to $\mathcal{G}$ where s_i is the i^{th} element in $\mathcal{S}$. Remove from L all multiples of t. Then continue with step 2).*

(5) *Otherwise $(v_1, \ldots, v_s) \neq (0, \ldots, 0)$, so append $(v_1, \ldots, v_s)$ as a new row to $\mathcal{M}$ and $t - \sum_i a_i s_i$ as a new element to $\mathcal{S}$. Add t to O, and add to L those elements of $\{x_1 t, \ldots, x_n t\}$ which are neither multiples of an element of L nor of $\mathrm{LT}_\sigma(\mathcal{G})$. Continue with step 2).*

This is an algorithm which returns $(\mathcal{G}, O)$ such that $\mathcal{G}$ is the reduced σ-Gröbner basis of $\mathcal{I}(\mathbb{X})$ and $O = \mathbb{T}^n \setminus \mathrm{LT}_\sigma\{\mathcal{I}(\mathbb{X})\}$.

The BM Algorithm II

A small alteration of this algorithm allows us to compute the separators of $\mathbb{X}$ as well.

Corollary

In the setting of the theorem, replace step 2) by the following instruction.

(2') *If* $L = \emptyset$ *then row reduce* $\mathcal{M}$ *to a diagonal matrix and mimic these row operations on the elements of* $\mathcal{S}$ *(considered as a column vector). Next replace* $\mathcal{S}$ *by* $\mathcal{M}^{-1}\mathcal{S}$ *, return the triple* $(\mathcal{G}, O, \mathcal{S})$ *, and stop. If* $L \neq \emptyset$ *, choose the term* $t = \min_\sigma(L)$ *and delete it from* L *.*

The resulting sequence of instructions defines again an algorithm. It returns a triple $(\mathcal{G}, O, \mathcal{S})$ *such that* $\mathcal{G}$ *is the reduced* σ *-Gröbner basis of* $\mathcal{I}(\mathbb{X})$ *, such that* $O = \mathbb{T}^n \setminus \mathrm{LT}_\sigma\{\mathcal{I}(\mathbb{X})\}$ *, and the tuple* $\mathcal{S}$ *contains the separators of* p_i *from* $\mathbb{X} \setminus p_i$ *for* $i = 1, \ldots, s$ *.*

Remarks

- More details on BM can be found in the two papers (see [ABKR] and [AKR]).
- Using Buchberger-Möller algorithm and its corollary it is possible to compute ideals of points and to solve the problem of interpolation.
- What happens if the data are not exact?
- Approximate versions of the above results should be able to construct sets of polynomials which almost vanish at $\mathbb{X}$, almost separators and almost interpolators.
- Of course new problems of stability arise in this context.

Bases of P/I

- In the last section we pointed to questions of stability. We have seen that methods for solving polynomial systems inevitably feature problems of approximation. On the other hand, problems of the same type arise if the data are not exact.
- If we go back for a moment to the eigenvalue method, we remember that it requires multiplication matrices which, in turn, require the choice of a basis of $A = P/I$ as a K-vector space.
- One way to get a basis of A is via Gröbner bases. If σ is a term ordering on $\mathbb{T}^n$ and $G = \{f_1, \dots, f_s\}$ is a σ-Gröbner basis of I, then $\mathrm{LT}_\sigma(I) = (\mathrm{LT}_\sigma(f_1), \dots, \mathrm{LT}_\sigma(f_s))$. We know that the residue classes of the elements of $\mathbb{T}^n \setminus \mathrm{LT}_\sigma(I)$ form a K-basis of A.
- Therefore, once a σ-Gröbner basis of I is computed, a basis is available. If we change σ we may get different bases of A, and a first question arises here.
- **Question 1** Are these the only bases?

  And in connection with the stability issue, the next question is.
- **Question 2** What are the most stable bases?

Filtration cycle I

Before answering these question, we make a digression into the realm of statistics (see Tutorial 92 of [KR2]).

A classical real world problem reads as follows. A number of similar chemical plants had been successfully operating for several years in different locations. In a newly constructed plant the filtration cycle took almost twice as long as in the older plants. Seven possible causes of the difficulty were considered by the experts.

1. The water for the new plant was different in mineral content.
2. The raw material was not identical in all respects to that used in the older plants.
3. The temperature of filtration in the new plant was slightly lower than in the older plants.
4. A new recycle device was absent in the older plants.
5. The rate of addition of caustic soda was higher in the new plant.
6. A new type of filter cloth was being used in the new plant.
7. The holdup time was lower than in the older plants.

Filtration cycle II

- These causes lead to seven variables $x_1, \dots, x_7$.
- Each of them can assume only two values, namely *old* and *new* which we denote by 0 and 1, respectively. All possible combinations of these values form the full design $D = \{0,1\}^7 \subseteq \mathbb{A}^7(\mathbb{Q})$.
- Its vanishing ideal is $\mathcal{I}(D) = (x_1^2 - x_1,\, x_2^2 - x_2, \dots, x_7^2 - x_7)$ in the polynomial ring $\mathbb{Q}[x_1, \dots, x_7]$.
- Our task is to identify an unknown function $\bar{f} : D \longrightarrow K$, namely the length of a filtration cycle. This function is called the model, since it is a mathematical model of the quantity which has to be computed or optimized.

Filtration cycle III

- In order to fully identify it, we would have to perform $128 = 2^7$ cycles. This is impracticable since it would require too much time and money.
- On the other hand, suppose for a moment that we had conducted all experiments and the result was $\bar{f} = a + b\,x_1 + c\,x_2$ for some $a, b, c \in \mathbb{Q}$. At this point it becomes clear that we have wasted many resources. Had we known in advance that $\bar{f}$ is given by a polynomial having only three unknown coefficients, we could have identified them by performing only *three* suitable experiments! Namely, if we determine three values of $a + b\,x_1 + c\,x_2$ and the associated matrix of coefficients is invertible, we can easily find a, b, c by solving a system of three linear equations in these three unknowns.
- However, *a priori* one does not know that the answer has the shape indicated above. In practice, one has to make some guesses, perform well-chosen experiments, and possibly modify the guesses until the process yields the desired answer.
- In the case of the chemical plant, it turned out that only x_1 and x_5 were relevant for identifying the model.

Fractions

- Motivated by this example, we introduce a piece of notation.
- Let K be a field. For $i = 1, \ldots, n$, let $\ell_i \geq 1$ and $D_i = \{a_{i1}, a_{i2}, \ldots, a_{i\ell_i}\} \subseteq K$. Then we say that the full design $D = D_1 \times \cdots \times D_n \subseteq \mathbb{A}^n(K)$ has levels $(\ell_1, \ldots \ell_n)$.
- The polynomials $f_i = (x_i - a_{i1}) \cdots (x_i - a_{i\ell_i})$ with $i = 1, \ldots, n$ generate the vanishing ideal $\mathcal{I}(D) \subseteq P$ of D. They are called the canonical polynomials of D. For any term ordering σ on $\mathbb{T}^n$, the canonical polynomials are the reduced σ-Gröbner basis of $\mathcal{I}(D)$.
- Thus the order ideal

$$O_D = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid 0 \leq \alpha_i < \ell_i \text{ for } i = 1, \ldots, n\}$$

  is canonically associated to D and represents a K-basis of $P/\mathcal{I}(D)$.
- Our main task is to identify an unknown function $\bar{f} : D \longrightarrow K$ called the model. We want to choose a fraction $F \subseteq D$ that allows us to identify the model if we have some extra knowledge about the form of $\bar{f}$.

Inverse problem

- Given a full design D and an order ideal $O \subseteq O_D$, which fractions $F \subseteq D$ have the property that the residue classes of the elements of O are a K-basis of $P/\mathcal{I}(F)$?
- It is called the inverse problem in DoE (Design of Experiments).
- The solution of this problem already in year 2001 (see [CR]) showed the importance of a new notion in commutative algebra, that of border basis.
- It was a first answer to **Question 1**. Let us move to **Question 2** by considering the following example.