Tutorial 2: Faster! Faster! The FGLM Speed-Up

In the following we want to improve the `LexSolver(...)` of Tutorial 1 in the following way: we compute *one* Gröbner basis of the ideal $I = \langle f_1, \ldots, f_s \rangle$ in the very beginning. (Usually, the term ordering $\sigma =$ `DegRevLex` should be used for this purpose.) Whenever we need a Gröbner basis of I with respect to a different term ordering, we apply the FGLM Algorithm to compute it using linear algebra. If we have to change the coordinate system to bring the ideal into normal x_n-position, we avoid the computation of a `Lex`-Gröbner basis with the same technique. Let us begin with a little bit of theory.

a) **(The FGLM Algorithm)**
Let K be a field, let $P = K[x_1, \ldots, x_n]$, let $I \subseteq P$ be a zero-dimensional polynomial ideal, let σ, τ be term orderings on $\mathbb{T}^n$, and let $G = \{g_1, \ldots, g_s\}$ be a σ-Gröbner basis of I. Consider the following instructions.

1. Let $\mathcal{O} = \emptyset$, $H = \emptyset$, and $L = \emptyset$.
2. Determine the smallest term t w.r.t. τ such that $t \notin \langle \mathrm{LT}_\tau(H) \rangle$ and $t \notin \mathcal{O}$. If there is no such term, return $(H, \mathcal{O})$ and stop.
3. Compute $h = \mathrm{NR}_{\sigma,G}(t)$ and check whether $h \in \langle L \rangle_K$.
4. If $h \in \langle L \rangle_K$ then write $h = a_1 p_1 + \cdots + a_m p_m$ with $a_i \in K$ and $p_i \in L$. Let t_i denote the term for which $p_i = \mathrm{NR}_{\sigma,G}(t_i)$. Append the polynomial $p = t - a_1 t_1 - \cdots - a_m t_m$ to H and continue with step (2).
5. If $h \notin \langle L \rangle_K$ then append h to L and t to $\mathcal{O}$. Continue with step (2).

Show that this is an algorithm which computes a τ-Gröbner basis H of I and the order ideal $\mathcal{O} = \mathbb{T}^n \setminus \mathrm{LT}_\sigma\{I\}$.

b) Prove that one can implement step (2) of the FGLM Algorithm in the following way.

1. Compute the border $\partial\mathcal{O}$.
2. Compute the set $C = \{t \in \partial\mathcal{O} \mid t' \nmid t \text{ for } t' \in \partial\mathcal{O} \setminus \{t\}\}$.
3. Compute the set $S = C \setminus \mathrm{LT}_\tau\{H\}$.
4. If $S = \emptyset$ then return $(H, \mathcal{O})$ and stop. Otherwise, let $t = \min_\tau(S)$.

c) Implement the FGLM Algorithm in a CoCoA function `FGLM(...)`.

Hints: Keep the elements of L interreduced, so that they have distinct leading terms. In this way it is easy to decide whether $h \in \langle L \rangle$ and to find the representation in (4).

d) Apply your function `FGLM(...)` to the zero-dimensional ideals in Tutorial 1 with $\sigma =$ `DegRevLex` and $\tau =$ `Lex`. Compare your results to the results of the built-in ApCoCoA function `FGLM5(...)`.

e) Assume that $I \subseteq P$ is a zero-dimensional radical ideal which is not in normal x_n-position. Let τ be a term ordering such that $x_n >_\tau x_i$ for $i = 1, \ldots, n-1$, and let $G = \{g_1, \ldots, g_s\}$ be a τ-Gröbner basis of I. Let $(c_1, \ldots, c_{n-1}) \in K^{n-1}$, and let $\varphi : P \longrightarrow P$ be the linear change of coordinates such that $x_i \mapsto x_i$ for $i = 1, \ldots, n-1$ and $x_n \mapsto x_n - c_1 x_1 - \cdots - c_{n-1} x_{n-1}$. Show that $\varphi(G) = \{\varphi(g_1), \ldots, \varphi(g_s)\}$ is a τ-Gröbner basis of $\varphi(I)$.

Hint: Prove that $\mathrm{LT}_\tau(\varphi(g_i)) = \mathrm{LT}_\tau(g_i)$ for all i.

f) Now implement a CoCoA function `FGLMSolver(...)` which replaces the Gröbner basis computations necessary for `LexSolver(...)` as follows.

1. Compute the `DegRevLex`-Gröbner basis of I.
2. Use FGLM to determine the elimination polynomials $g_i \in K[x_i]$ such that $\langle g_i \rangle = I \cap K[x_i]$.
3. If it is necessary to perform a linear change of coordinates, use FGLM to compute a Gröbner basis w.r.t. a term ordering τ which satisfies $x_n >_\tau x_i$ for $i = 1, \ldots, n-1$.
4. Finally, use FGLM again to determine the desired `Lex`-Gröbner basis.

g) Apply your `FGLMSolver(...)` to the examples in Tutorial 1. Can you find examples for which there is a significant difference in the timings?