EXPERIMENTS IN COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY

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1. Introductory Remarks

This is an extended set of lecture notes for the course given by the author at the CoCoA school in Sardegna in May, 2005. These notes are not intended to be exhaustive. Instead, the goal is to present the student with an overview of certain areas of algebraic geometry and commutative algebra which are extremely active, and to start seeing how almost all of the constructions and theory can be carried out on the computer algebra program CoCoA. For this reason, a fundamental part of this course is the series of tutorials run by Martin Kreuzer. These notes contain some exercises that may supplement those tutorials or simply give the student a way of testing his or her understanding of the material. Of course it is necessary to look up the topics in the literature in order to get a more detailed treatment of this material. What we have here is only a rough overview. It is hoped that it is enough for the student to whet his or her appetite, and that there are enough details here for the student to at least begin running some experiments on the computer and to write CoCoA programs on these subjects.

The three best references for this course (including the tutorials) are

- Kreuzer-Robbiano, *Computational Commutative Algebra, I & II*
- Migliore, *Introduction to Liaison Theorey and Deficiency Modules*
- Schenck, *Computational Algebraic Geometry*

See the references at the end for the details.

Goal of course: To start understanding how one uses computer algebra programs (in our case, CoCoA) as a tool for research. In a sense, Commutative Algebra and Algebraic Geometry are becoming experimental sciences!!

- What kinds of problems lend themselves to being helped by experiments? (Nearly all...)

- What kinds of experiments do you try to perform for a given problem? Sometimes the hardest part is figuring out how to run an experiment that will shed light on the problem.
- How do you go about carrying out these experiments?

Structure of Course: My lectures will be more on the first two questions. The material from the lectures will be used in the tutorials to get a hands-on feeling for how to start working on your own. We'll go quickly over a lot of material, with few proofs. We want to get started quickly on the computer!

The material for much of the course will be loosely centered on questions around the topic of Gorenstein algebras. Unfortunately we have a broad range of backgrounds here, so for many there will be some review, more for some, less for others. Much of it will be overview with references.

Truth in Advertising Statement: I am not at all an expert in using CoCoA. I learned how to work in `Macaulay (Classic)`, and am only now starting to learn CoCoA. But I and my co-authors (notably Chris Peterson) have used computer experiments in probably all of the last 30 papers or so that I have written, and many of them would not have been written without the computer-aided experiments.

However, Martin Kreuzer will be running the tutorials, and he is one of the world experts in using computer algebra programs in general, and CoCoA in particular.

How does a computer help in research?

- **Playing around**, you notice patterns and try to explain them or make conjectures. For instance, the theory of Buchsbaum-Rim sheaves (e.g. [60]) arose in this way, thanks to Chris Peterson's experiments. This will be described briefly below.
- **Test conjectures/ideas.** It often requires a lot of theoretical work to figure out the right example to run.
- **Produce interesting examples** of theorems that you have proven, by careful choice. An example is in the paper, [38], of Huneke and Ulrich.
- **Produce exhaustive searches** that you can analyze and look to see if a statement is true or not, or look for patterns. Examples are searches done by Yong-Su Shin in [23] and by Chris Francisco in [18].

As you probably know already, or will certainly find out very soon (!), most of the basic constructions already exist as easy procedures in CoCoA (e.g. saturation, Hilbert function, minimal free resolutions). The art is to see how much farther you can go by writing programs to do what you want to study.

2. Hilbert functions

Definition 2.1. Let K be a field and let $P = K[x_0, \ldots, x_n]$ be the homogeneous polynomial ring. Let $\mathfrak{m}$ be the homogeneous maximal ideal, $\mathfrak{m} = (x_0, \ldots, x_n)$. Let I be a homogeneous ideal. The *saturation*, $\bar{I}$, of I is

$$\bar{I} = \{f \in P \mid \mathfrak{m}^t \cdot f \subset I \text{ for some } t > 0\}.$$

A homogeneous ideal $I \subset P$ is *saturated* if $I = \bar{I}$.

Remark 2.2. Saturated ideals are what define subschemes of projective space. We will usually start with saturated ideals, but many non-saturated ideals will be of interest to us. If you are not comfortable with the notion of a scheme, just think "saturated ideal" wherever you see the word "scheme." There are infinitely many homogeneous ideals that define the same scheme in $\mathbb{P}^n$, and they all have the property that they have the same saturation, and any two (including the saturation) agree in all sufficiently large degrees.

Definition 2.3. Given a finitely generated graded module M, the *Hilbert function* of M is the function

$$\mathrm{HF}_M : \mathbb{Z} \to \mathbb{Z}$$

defined by $t \mapsto \dim_K(M_t)$.

For us the most important special case is when $M = P/I$ for some homogeneous ideal I, often saturated. Since both P and I are graded, we have

$$HF_{P/I}(t) = \dim_K P_t - \dim_K I_t.$$

Theorem 2.4 (Hilbert)**.** *Given a finitely generated graded module M with Hilbert function $\mathrm{HF}_M(t)$, there is a unique polynomial, $\mathrm{HP}_M(t)$ with rational coefficients such that $\mathrm{HP}_M(t) = \mathrm{HF}_M(t)$ for all $t \gg 0$. We have $\deg \mathrm{HP}(t) = \dim Z(\mathrm{Ann} M)$, where Z denotes the zero set in $\mathbb{P}^n$ of a homogeneous ideal.*

(See Hartshorne [30] Theorem I.7.5 for the proof.)

Remark 2.5. When $M = R/I$ for a saturated ideal I defining a scheme V, then the coefficients of HP have geometric significance. In this case we often write $\mathrm{HP}_V(t)$ for $\mathrm{HP}_{P/I}(t)$.

For example, when V is a curve, we have

$$\mathrm{HP}_V(t) = (\deg V)t - p_a(V) + 1,$$

where $p_a(V)$ denotes the arithmetic genus.

Example 2.6. (1) If $M = P = K[x_0, x_1, \dots, x_n]$, we know that

$$\mathrm{HF}_P(t) = \dim P_t = \binom{t+n}{t}.$$

(2) If V is a zero-dimensional scheme of degree d then $\mathrm{HP}_V(t) = d$ for all $t \gg 0$.
(3) If V is a smooth rational curve of degree d then $\mathrm{HP}_V = dt+1$ (regardless of which projective space V lives in).
(4) If V is a smooth rational curve of degree 6 in $\mathbb{P}^3$ lying on a quadric surface then one can check that

$$HF_V(t) = \begin{cases} 1, & \text{if } t = 0; \\ 4, & \text{if } t = 1; \\ 9, & \text{if } t = 2; \\ 16, & \text{if } t = 3; \\ 6t+1, & \text{if } t \geq 4; \end{cases}$$

(5) If V is a smooth rational curve of degree 6 in $\mathbb{P}^3$ not lying on a quadric surface but lying on a unique cubic (this is not the only possibility, but it's the only one we'll discuss) then

$$HF_V(t) = \begin{cases} 1, & \text{if } t = 0; \\ 4, & \text{if } t = 1; \\ 10, & \text{if } t = 2; \\ 6t+1, & \text{if } t \geq 3; \end{cases}$$

Exercise 1. (but I don't know the full answer): If V is a smooth rational curve in $\mathbb{P}^3$ not lying on a quadric, how many independent cubics can it lie on? I.e. what are the possibilities for $\dim(I_V)_3$?

Example 2.7. If $P = K[x_0, x_1, x_2]$ and $I = (x_0^2, x_1^3, x_2^6)$ then what can we say about the Hilbert function? Let $\mathfrak{m} = (x_0, x_1, x_2)$. Of course from now on we will never do such a thing by hand, but let's see what's involved. And let's be happy that the ideal is (x_0^2, x_1^3, x_2^6) and not

$(330206918x^2 + 2283577463xy + 2129104241y^2 - 2703411994xz + 2216678917yz - 1581465340z^2,$

$-478900145x^3 + 3323470647x^2y - 612426241xy^2 + 3469926795y^3 - 2210565765x^2z - 1834910978xyz + 3396757959y^2z + 1258111242xz^2 + 2398521289yz^2 - 1117801112z^3,$

$-3955911391x^6 - 2025890740x^5y + 3202146114x^4y^2 + 3900633001x^3y^3 - 2371292351x^2y^4 - 2057716532xy^5 - 578365832y^6 + 1144565127x^5z + 1439020939x^4yz + 2060537173x^3y^2z - 3819178076x^2y^3z + 710288713xy^4z - 4037409953y^5z - 2976048620x^4z^2 - 2722226791x^3yz^2 + 3636460765x^2y^2z^2 - 3099411540xy^3z^2 + 97696535y^4z^2 - 234975492x^3z^3 + 1650277019x^2yz^3 + 2990538955xy^2z^3 - 2900041698y^3z^3 + 3145426837x^2z^4 + 4123904461xyz^4 - 4091701992y^2z^4 + 3700309012xz^5 - 3670841466yz^5 + 721152313z^6)$

If nothing else, such an exercise makes one really appreciate the computer!

- Since $x_0x_1^2x_2^5 \notin I$, we know that $(P/I)_8 \neq 0$. Once can check that this is a basis for $(P/I)_8$. On the other hand, one can check that $(P/I)_9 = 0$. So we have
$$\mathrm{HF}_{P/I}(8) = 1, \mathrm{HF}_{P/I}(9) = 0.$$
This tells us when we are finished. In general such things are done by considering *regularity*.
- Let's start from the other side. Clearly
$$\mathrm{HF}_{P/I}(0) = 1,\ \mathrm{HF}_{P/I}(1) = 3.$$
- Since there is exactly one generator of I of degree 2, we also easily get
$$\mathrm{HF}_{P/I}(2) = 6 - 1 = 5.$$
- Since
$$(x_0^2 \cdot \mathfrak{m}, x_1^3) = (x_0^3, x_0^2x_1, x_0^2x_2, x_1^3)$$
and these generators form a basis for I_3, we see that
$$\dim I_3 = (1) \cdot \binom{3}{2} + (1) \cdot \binom{2}{2},\ \text{so } \mathrm{HF}_{P/I}(3) = 10 - 4 = 6.$$
- Similarly,
$$(x_0^2 \cdot \mathfrak{m}^2, x_1^3 \cdot \mathfrak{m}) = (x_0^4, x_0^3x_1, x_0^3x_2, x_0^2x_1^2, x_0^2x_1x_2, x_0^2x_2^2, x_0x_1^3, x_1^4, x_1^3x_2),$$
so we have
$$\dim I_4 = \binom{4}{2} + \binom{3}{2} = 9,\ \text{so } \mathrm{HF}_{P/I}(4) = 15 - 9 = 6.$$
- In degree 5 something new happens. We try to compute as above, and get a first estimate
$$\dim I_5 = \binom{5}{2} + \binom{4}{2};$$
but this counts the monomial $x_0^2x_1^3$ twice! So in fact we have
$$\dim I_5 = \binom{5}{2} + \binom{4}{2} - 1 = 15,\ \text{so } \mathrm{HF}_{P/I}(5) = 21 - 15 = 6.$$
- We keep going in this way, but eventually we find we are subtracting *too much* and have to add some of it back. The final answer is that the Hilbert function of P/I is given by the vector
$$(1, 3, 5, 6, 6, 6, 5, 3, 1)$$
(and all other values are 0).

Exercise 2. Check the details of Example 2.7.

An important question is to characterize those functions which are Hilbert functions of some standard graded algebra (i.e. are of the form P/I for some I). This was answered by Macaulay, and will be the focus of one of our tutorials.

A very interesting subject in this regard is what happens when Macaulay's bound is achieved, i.e. when there is *maximal growth* of the Hilbert function. This has been studied by Gotzmann [28]. A cute study of the geometric consequences of these results can be found in [5].

3. Minimal Free Resolutions

All this adding and subtracting of the dimensions (and more!) in the last example is kept track of by free resolutions.

Definition 3.1. Let M be a finitely generated graded P-module. A *free resolution* of M is a long exact sequence

$$0 \to \mathbb{F}_k \to \mathbb{F}_{k-1} \to \cdots \to \mathbb{F}_2 \to \mathbb{F}_1 \to \mathbb{F}_0 \to M \to 0$$

where the $\mathbb{F}_i$ are finitely generated free modules and the maps are homogeneous of degree 0. (This means it preserves the degrees of domain and target, not that the entries of the matrices are constants!) This is the *minimal free resolution* if all the entries of all the matrices are either zero or forms of degree ≥ 1. If this resolution is minimal, we say that the *projective dimension* of M is k. We write $\mathrm{pd}(M) = k$.

If $\mathbb{F}_i = \bigoplus_{j \in \mathbb{Z}} P(-a_{ij})$, then we say that the a_{ij} are the *graded Betti numbers* of M. $\mathbb{F}_0$ keeps track of the minimal generators of M, $\mathbb{F}_1$ keeps track of the minimal first syzygies, etc. But when $M = P/I$, it turns out that $\mathbb{F}_0 = P$ and $\mathbb{F}_1$ keeps track of the minimal generators of I, etc. because you can split the long exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{F}_k \arrow[r] & \dots \arrow[r] & \mathbb{F}_2 \arrow[r] & \mathbb{F}_1 \arrow[rr] \arrow[dr] & & P \arrow[r] & P/I \arrow[r] & 0. \\
& & & & & I \arrow[ur] \arrow[dr] & & & \\
& & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Example 3.2. You can check that the minimal free resolution of $I = (x_0^2, x_1^3, x_2^6)$ is

$$0 \to P(-11) \xrightarrow{\phi_3} P(-5) \oplus P(-8) \oplus P(-9) \xrightarrow{\phi_2} P(-2) \oplus P(-3) \oplus P(-6) \xrightarrow{\phi_1} P \to P/I \to 0,$$

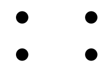
where

$$\phi_1 = \begin{bmatrix} x_0^2 & x_1^3 & x_2^6 \end{bmatrix}, \phi_2 = \begin{bmatrix} x_1^3 & x_2^6 & 0 \\ -x_0^2 & 0 & x_2^6 \\ 0 & -x_0^2 & -x_1^3 \end{bmatrix}, \phi_3 = \begin{bmatrix} x_2^6 \\ -x_1^3 \\ x_0^2 \end{bmatrix}$$

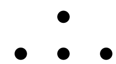
The exactness in each degree makes it easy to compute the Hilbert function and keep track of all the adding and subtracting we talked about.

In general knowing the graded Betti numbers gives you the Hilbert function for free, but not conversely – the graded Betti numbers contain more information.

Example 3.3. A set, Z_1, of four general points in $\mathbb{P}^2$



has Hilbert function $(1, 3, 4, 4, 4\dots)$, but so does a set, Z_2, of four points with three on a line:



But we can distinguish between these via the minimal free resolution, which picks up the fact that the first is a complete intersection (see below) and the second is not. The minimal free resolutions (ignoring the maps) are, respectively,

$$0 \to P(-4) \to P(-2)^2 \to I_{Z_1} \to 0,$$

and

$$0 \to \begin{array}{c} P(-4) \\ \oplus \\ P(-3) \end{array} \to \begin{array}{c} P(-2)^2 \\ \oplus \\ P(-3) \end{array} \to I_{Z_2} \to 0.$$

Note that what distinguishes them is a "ghost term" $P(-3)$, also called a "redundant term." Notice that such summands do not contribute to the computation of the Hilbert function.

Ghost terms (or more precisely, their absence) have appeared in several very interesting conjectures. We will discuss these in greater detail below, but we mention them now. First we have the Minimal Resolution Conjecture:

Conjecture 3.4 (Lorenzini, [47])**.** *The ideal of a general set of points in $\mathbb{P}^n$ has a minimal free resolution with no ghost terms.*

This conjecture was disproven by Eisenbud and Popescu [17] a few years ago. However, special cases have either been proven (e.g. the Cohen-Macaulay type conjecture for the end of the resolution, proven by Lauze [45]) or remain open (e.g. the Ideal Generation conjecture for the beginning of the resolution).

Another conjecture that has appeared is for an ideal of generally chosen forms of arbitrary degree. Note first that sometimes Koszul relations force ghost terms. For instance, even for generally chosen forms of degrees 4,4,8 in three variables (say), there is forced to be a generator of degree 8 and a Koszul first syzygy of degree 8 coming from the two generators of degree 4. If you add generators of degrees that are not too small or too large, this problem remains. So we have the Thin Resolution Conjecture:

Conjecture 3.5 (Iarrobino, [39])**.** *An ideal of generally chosen forms has a minimal free resolution with no ghost terms apart from those forced by Koszul syzygies.*

This was also disproven [54]. See also [55], [56]. However, again special cases remain, and in both cases it is probably true that "most of the time" the conjecture is true, and it remains to make clear what "most of the time" means. In the tutorials we will also discuss one of the special cases that is open, which is an Artinian version of the Minimal Resolution Conjecture. This is connected to the Fröberg Conjecture, which we will discuss below.

For now, we address the question of existence of minimal free resolutions.

Theorem 3.6 (Hilbert Syzygy Theorem)**.** *If M is a finitely generated graded P-module then M has a (finite) minimal free resolution, and it must hold that $k \leq n+1$.*

In fact we have something stronger:

Theorem 3.7 (Auslander-Buchsbaum)**.** $\mathrm{pd} M + \mathrm{depth}\ M = \mathrm{depth}\ P = n+1$.

Exercise 3. Verify that in Example 3.2, the Auslander-Buchsbaum theorem gives $3+0=3$.

Definition 3.8. The quotient ring P/I is *Cohen-Macaulay* if

$$\mathrm{depth}\ P/I = \dim P/I,$$

where dim refers to the Krull dimension. A subscheme V of $\mathbb{P}^n$ is *arithmetically Cohen-Macaulay (ACM)* if P/I_V is a Cohen-Macaulay ring.

Corollary 3.9. *If P/I is Cohen-Macaulay of Krull dimension $r+1$ then $\mathrm{pd} P/I = n-r$. In particular, if $V \subset \mathbb{P}^n$ is arithmetically Cohen-Macaulay then $\mathrm{pd} P/I_V = \mathrm{codim}\ V$. Otherwise $\mathrm{pd} P/I_V > \mathrm{codim}\ V$.*

Definition 3.10. If P/I is Cohen-Macaulay of Krull dimension $r+1$ then the *Cohen-Macaulay type* of P/I is the rank of the last free module, $\mathbb{F}_{n-r}$.

A special case is when the number of minimal generators of I_V is equal to the codimension of V (i.e. height of I_V). Then V is called a *complete intersection*. Our more algebraically inclined friends call I_V a *regular sequence*. Many invariants of V depend only on the degrees of the generators of I_V, called the *type* of the complete intersection. (This should not be confused with the Cohen-Macaulay type.) For example, the two ideals at the beginning of Example 2.7 are complete intersections of the same type.

Some special properties of complete intersections are

- the degree is equal to the product of the degrees of the generators;

- If P/I is finite dimensional as a K-vector space (i.e. if it is *Artinian*) then the Hilbert function of P/I is symmetric (looking only at the non-zero entries). We saw this above in the case of (x_0^2, x_1^3, x_2^6), where the Hilbert function was $(1, 3, 5, 6, 6, 6, 5, 3, 1)$. In the case of the four points, where the Hilbert function was $(1, 3, 4, 4, 4, \dots)$, it is not symmetric, but notice that the *first difference* is $(1, 2, 1)$, which is symmetric. This will become clearer in the next section.
- The minimal free resolution is well-known, and is called the *Koszul resolution*. An example was given above in the case of (x_0^2, x_1^3, x_2^6). It always turns out (for complete intersections) that the last free module in the Koszul resolution is $P(-k)$, where k is the sum of the degrees of the minimal generators. It also turns out that the resolution is self-dual (up to twist). We will generalize this shortly.

In many situations (as we will see), we are more interested in the ranks and twists of the modules $\mathbb{F}_i$ (i.e. in the graded Betti numbers) than in the maps ϕ_i. This information is kept track of in the *Betti diagram*: if $\mathbb{F}_i = \bigoplus_{j \in \mathbb{Z}} P(-a_{ij})$ then we record this information with the following diagram (where we assume that the generators are all of non-negative degree):

$$\begin{array}{c|cccc} & b_0 & b_1 & b_2 & \dots \\ \hline 0 & a_{0,0} & a_{1,1} & a_{2,2} & \dots \\ 1 & a_{0,1} & a_{1,2} & a_{2,3} & \dots \\ 2 & a_{0,2} & a_{1,3} & a_{2,4} & \dots \\ 3 & a_{0,3} & a_{1,4} & a_{2,5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

where $a_{i,j}$ gives the number of copies of $P(-j)$ in $\mathbb{F}_i$, and b_i is the rank of $\mathbb{F}_i$, and is equal to $\sum_j a_{i,j}$. By convention, we often write a dash "-" in place of a 0.

Example 3.11. If $I = (x_0^2, x_1^3, x_2^6)$ then the Betti diagram for P/I is

	1	3	3	1
0	1	-	-	-
1	-	1	-	-
2	-	1	-	-
3	-	-	1	-
4	-	-	-	-
5	-	1	-	-
6	-	-	1	-
7	-	-	1	-
8	-	-	-	1

4. Hyperplane sections

An important tool (this is a huge understatement) in algebraic geometry and commutative algebra is that of taking hyperplane sections. Geometrically, we intersect V with a general hyperplane ("general" can mean different things, depending on the situation) to obtain $V \cap H$. We sometimes call this the *geometric hyperplane section*.

Definition 4.1. Let $I \subset$ be a saturated ideal defining a closed subscheme $V \subset \mathbb{P}^n$. Let L be a linear form such that L is not a zero-divisor on P/I. Then the *algebraic hyperplane section* is the ideal

$$J := \frac{I + (L)}{(L)} \subset R := P/(L).$$

The saturation of J is the ideal $I_{V\cap H}$ of the geometric hyperplane section.

Note that J may or may not be saturated. Its saturation depends on some cohomological conditions, namely the vanishing of

$$\bigoplus_{t\in\mathbb{Z}} H^1(\mathbb{P}^n, \mathcal{I}_V(t)).$$

This module is often called the *deficiency module*, and in the case of curves it is also called the *Hartshorne-Rao module*. It is not hard to check that V is ACM if and only if $H^i(\mathbb{P}^n, \mathcal{I}_V(t)) = 0$ for all t and all $1 \leq i \leq \dim V$. In the case of curves, we only have to worry about $i = 1$.

Example 4.2. Let V_1 be the union of two lines in $\mathbb{P}^3$ meeting in a point. Then V_1 is a plane curve of degree 2, and it is a complete intersection of type $(1, 2)$. The geometric hyperplane section consists of two points, which form a complete intersection of type $(1, 2)$ in the plane defined by L (i.e. in the ring $P/(L)$). It is not hard to check that J_1 is saturated.

Let V_2 be the union of two disjoint lines in $\mathbb{P}^3$. Then V_2 does not lie in any plane, i.e. I_{V_2} does not contain any linear forms. So J_2 does not contain any linear form (as an ideal in $P/(L)$). But $V_2 \cap H$ is again a set of two points, which does lie on a line. So J_2 is not saturated.

The difference between V_1 and V_2 is that V_1 is ACM while V_2 is not.

Proposition 4.3. *Let I be a homogeous ideal such that P/I is Cohen-Macaulay of Krull dimension $r \geq 1$. Then the following hold:*

1. *I is saturated.*
2. *Let J be the algebraic hyperplane section, $J = \frac{I+(L)}{(L)} \subset P/(L) = R$, as above. Then R/J is again Cohen-Macaulay, now of Krull dimension $r - 1$.*
3. *The ideal J has the same Betti diagram in the ring R as I does in P.*
4. *The Hilbert function of R/J is the first difference of that of P/I.*

(5) *In particular, the degree is preserved. This is true even without the Cohen-Macaulay assumption when $r \geq 2$, but it's also true for $r = 1$ in the Cohen-Macaulay case. (If R/J is Artinian then by "degree" we just mean the vector space dimension of R/J.)*

(6) *If the Krull dimension of P/I is ≥ 2 then J is saturated, as an ideal in R. Of course if R/J is Artinian then J is not saturated.*

Remark 4.4. (a) If P/I is Artinian then depth $P/I = \dim P/I = 0$ (where "dim" refers to the Krull dimension), so P/I is automatically Cohen-Macaulay.

(b) If P/I is the coordinate ring of a zero-dimensional scheme (i.e. the Krull dimension of P/I is 1 and I is saturated, so depth $P/I \geq 1$) then in fact depth $P/I = 1$ and P/I is again automatically Cohen-Macaulay.

Exercise 4. If R/J is Artinian then what is the saturation of J?

Definition 4.5. If P/I is Cohen-Macaulay of Krull dimension r then the *Artinian reduction* of P/I is the ring obtained by performing the process of taking algebraic hyperplane sections a total of r times. This can be accomplished in one step by replacing L by $(L_1, \ldots, L_r)$. The Artinian reduction is a Cohen-Macaulay ring of Krull dimension 0, and R/J is finite-dimensional as a K-vector space. The Hilbert function of R/J is called the *h-vector* of P/I.

A very interesting question is to determine the extent to which the converse of (2) is true in Proposition 4.3. To begin, we have the following (see for instance [38] or [51]).

Proposition 4.6. *Let $V \subset \mathbb{P}^n$ be a subscheme of dimension $r \geq 2$ (as a subscheme of $\mathbb{P}^n$). If the general* geometric *hyperplane section is ACM then V is ACM.*

Of course if V is a curve then its geometric hyperplane section is a zero-dimensional scheme, which is ACM. But we have seen that not all curves are ACM. A good deal of work has centered on the question of finding conditions on the *general* geometric hyperplane section that guarantee that V is in fact ACM. There are too many papers to cite; the most recent is the paper [27] of Gorla, and we refer to that paper for the other references.

More generally, there is the whole "lifting" problem:

- What Artinian rings are the Artinian reduction of the coordinate ring of a reduced set of points? This is completely open. In the tutorial we will discuss one answer, namely *distractions*, or *liftings* of Artinian monomial ideals. A *distraction* of an Artinian monomial ideal gives the homogeneous saturated ideal of a finite, reduced set of points whose Artinian reduction is the original monomial ideal. The idea goes back at least to Hartshorne [31], and was extended in [57].

- What finite Hilbert functions are the Hilbert function of the Artinian reduction of a reduced set of points? (This was answered by Geramita-Maroscia-Roberts [24].) The answer is that the only condition is Macaulay's growth condition!!! This is shown via distractions.
- What finite Hilbert functions are the Hilbert function of the Artinian reduction of a reduced set of points with given properties? This has been studied in the case of *Gorenstein* and *level* rings, and points with the Uniform Position Property (see below on page 12), but is largely open.

5. Codimension Two ACM schemes

If V is an ACM subscheme of $\mathbb{P}^n$ codimension two, we know that the minimal free resolution of R/I_V has the form

$$0 \to \mathbb{F}_2 \xrightarrow{A} \mathbb{F}_1 \to R \to R/I_V \to 0.$$

The matrix A is called the *Hilbert-Burch matrix* of V (or of I_V). Because the alternating sum of the ranks is zero, we see that A has to be a $(t+1) \times t$ homogeneous matrix.

Theorem 5.1 (Hilbert-Burch)**.** *I_V is minimally generated by the maximal minors of A. Conversely, if A is a homogeneous matrix whose ideal, I, of maximal minors define a scheme of codimension two, then I is saturated, and in fact P/I is Cohen-Macaulay, and after a change of basis, A appears as the matrix in the minimal free resolution.*

If it seems like practically every theorem in these notes has Hilbert's name on it, you're not too far off!

Without going into too much detail, just about every theorem about codimension two ACM schemes is centered around this theorem. For instance, when we talk about liaison, the fact that we know so much about the liaison properties of ACM schemes of codimension two is due to this theorem.

An interesting problem is to describe the possible Hilbert functions of codimension two ACM subschemes, and without loss of generality we might as well describe the possible Hilbert functions of points in $\mathbb{P}^2$. This will be treated in one of the tutorials, as a special case. (In fact, we will discuss what happens for points in any projective space.)

Exercise 5. Why is this "without loss of generality?"

What is not treated in the tutorials is the fact that much more is known for points in $\mathbb{P}^2$. In particular, the following have been studied:

- What are the possible Hilbert functions for sets of points in $\mathbb{P}^2$ with the Uniform Position Property (UPP)? A set of points is said to have the *Uniform Position*

Property if all subsets of the same cardinality have the same Hilbert function. For instance, in Example 3.3, the first set of points has UPP while the second set does not. The necessary and sufficient condition is that the Hilbert function has to be of *decreasing type*. This is a property of the h-vector, and it says that after the h-vector reaches its maximum and starts decreasing, it has to be strictly decreasing until it reaches zero. One place to read about this is [25], which describes the history of the problem and the different ways in which the problem has been answered in the literature.

- It was noted above that the possible Hilbert functions of reduced sets of points in $\mathbb{P}^n$ are known (via their Artinian reductions). What happens if the growth from any particular degree to the next is maximal? This is actually more interesting at the Artinian level. In $\mathbb{P}^2$ this is completely understood, and it is due to Davis [13]. Some results in the same direction for higher projective space can be found in [5]. Davis' result says, basically, that if the Artinian reduction, A, has maximal growth of the Hilbert function from degree d to $d+1$, say, then the component in degree $d+1$ has a GCD of degree equal to $HF_A(d)$. In this context, it turns out that "maximal growth" means that $HF_A(d) = HF_A(d+1)$. In the case of reduced points, Davis also gives geometric consequences about how many points must lie on the curve defined by the GCD.

6. Gorenstein rings

In this section we will discuss a natural extension of the notion of a complete intersection, which has been extremely important in the literature.

Definition 6.1. If I is an ideal such that P/I is Cohen-Macaulay and the minimal free resolution of P/I ends with a free module $\mathbb{F}_k$ of rank 1 (i.e. if P/I has Cohen-Macaulay type 1) then P/I is a *Gorenstein ring*. If $V \subset \mathbb{P}^n$ is a subscheme such that P/I_V is Gorenstein, we say that V is *arithmetically Gorenstein*.

Remark 6.2. While we will not discuss this too much, it is worth noting that a generalization of this is the notion of a *level ring*, which is simply defined by the property that the last free module $\mathbb{F}_r$ does not necessarily have rank one, but all summands of $\mathbb{F}_r$ have the same twist. An example is a rational normal curve.

Proposition 6.3. *If P/I is Gorenstein of Krull dimension $r+1$ then*

(1) *The minimal free resolution of P/I is self-dual up to twist;*
(2) *The Hilbert function of the Artinian reduction of P/I is symmetric;*
(3) The canonical module $\mathrm{Ext}_R^{n-r}(P/I, R)$ *is isomorphic to a twist of P/I.*

The following useful result is from [14]. It was generalized to the non-reduced case by Kreuzer [41]. (Ask him about it if you are interested!)

Theorem 6.4 (Davis-Geramita-Orecchia). *Let Z be a reduced zero-dimensional scheme of degree d. Then Z is arithmetically Gorenstein if and only if the Artinian reduction of R/I_Z has symmetric Hilbert function and Z has the Cayley-Bacharach property (i.e. all subsets of $d-1$ points have the same Hilbert function).*

Example 6.5.

1. The first example of a Gorenstein ring is any complete intersection.
2. Any $n+2$ points in general position in projective space $\mathbb{P}^n$ are arithmetically Gorenstein.
3. (Sums of linked ideals) Suppose that V_1 and V_2 are ACM subschemes of $\mathbb{P}^n$ of codimension c, with $1 \leq c \leq n$, having no common components, and whose union is arithmetically Gorenstein. Then $J := I_{V_1} + I_{V_2}$ is also arithmetically Gorenstein, but of codimension $c+1$. This follows from a standard and very useful trick called the *mapping cone*: consider the commutative diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & & 0 & & & & & \\
 & & \downarrow & & & \downarrow & & & & & \\
 & & P(-k) & & \mathbb{F}_c & \oplus & \mathbb{G}_c & & & & \\
 & & \downarrow & & & \downarrow & & & & & \\
 & & \mathbb{L}_{c-1} & & \mathbb{F}_{c-1} & \oplus & \mathbb{G}_{c-1} & & & & \\
 & & \downarrow & & & \downarrow & & & & & \\
 & & \mathbb{L}_{c-2} & & \mathbb{F}_{c-2} & \oplus & \mathbb{G}_{c-2} & & & & \\
 & & \downarrow & & & \downarrow & & & & & \\
 & & \vdots & & & \vdots & & & & & \\
 & & \downarrow & & & \downarrow & & & & & \\
 & & \mathbb{L}_2 & & \mathbb{F}_2 & \oplus & \mathbb{G}_2 & & & & \\
 & & \downarrow & & & \downarrow & & & & & \\
 & & \mathbb{L}_1 & & \mathbb{F}_1 & \oplus & \mathbb{G}_1 & & & & \\
 & & \downarrow & & & \downarrow & & & & & \\
0 & \rightarrow & I_{V_1} \cap I_{V_2} & \rightarrow & I_{V_1} & \oplus & I_{V_2} & \rightarrow & J & \rightarrow & 0 \\
 & & \downarrow & & & \downarrow & & & & & \\
 & & 0 & & & 0 & & & & &
\end{array}$$

   where the short exact sequence at the bottom is standard, and the vertical sequences are the minimal free resolutions of $I_{V_1} \cap I_{V_2}$ (which by hypothesis is Gorenstein), I_{V_1}, and I_{V_2}, respectively. Then the mapping cone gives the (not necessarily minimal) free resolution for the cokernel, J:

$$0 \rightarrow P(-k) \rightarrow \begin{array}{c} \mathbb{L}_{c-1} \\ \oplus \\ \mathbb{F}_c \oplus \mathbb{G}_c \end{array} \rightarrow \begin{array}{c} \mathbb{L}_{c-2} \\ \oplus \\ \mathbb{F}_{c-1} \oplus \mathbb{G}_{c-1} \end{array} \rightarrow \cdots \rightarrow \begin{array}{c} \mathbb{L}_1 \\ \oplus \\ \mathbb{F}_2 \oplus \mathbb{G}_2 \end{array} \rightarrow \mathbb{F}_1 \oplus \mathbb{G}_1 \rightarrow J \rightarrow 0.$$

   One checks that the minimal free resolution of P/J has to have length at least $c+1$ (since V_1 and V_2 have no common component, so their intersection has codimension at least $c+1$). Hence the intersection is ACM of Cohen-Macaulay type 1, so it is Gorenstein.

(4) A beautiful example of a powerful result that came about from playing with computer algebra programs (in this case with `macaulay`) was carried out by Chris Peterson, in discovering the class of subschemes arising as *sections of Buchsbaum-Rim sheaves*. In particular, for Buchsbaum-Rim sheaves of odd rank on $\mathbb{P}^3$, the ideals produced are not saturated, but their saturation is arithmetically Gorenstein!! (This is a special case of the larger theory.) Of course after you discover a pattern on the computer, your next job is to prove it!! This was carried out in [63] and in [60].

A simple example of this procedure is the following. Let $P = K[x_0, x_1, x_2, x_3]$. Consider the 1×4 matrix $A = [x_0, x_1, x_2, x_3]$. This is the presentation matrix for the ring $P/\mathfrak{m} \cong K$, and we have the minimal free resolution

$$0 \to P(-4) \to P(-3)^4 \to P(-2)^6 \xrightarrow{B} P(-1)^4 \xrightarrow{A} P \to P/\mathfrak{m} \to 0.$$

B is a 4×6 homogeneous matrix, given by the Koszul resolution (since $\mathfrak{m}$ is a complete intersection). Take a general linear combination of the columns of B with coefficients that are homogeneous polynomials of any degree ≥ 1. This will be a 4×1 matrix of homogeneous polynomials. These four polynomials define an ideal I. It turns out that I is not saturated, but its saturation is a Gorenstein ideal with five generators, and its minimal free resolution can be written from the given information.

Exercise 6. Show how Example 6.5 (2) follows from Theorem 6.4.

Exercise 7. Write a program that computes the ideal described in Example 6.5 (4), for coefficients of arbitrary degree d. Look up the references and write a program that works for any Buchsbaum sheaf on $\mathbb{P}^3$, or on $\mathbb{P}^n$.

Very little is known about Gorenstein rings, but in low codimension a fair amount is known.

Theorem 6.6. *In codimension two, every Gorenstein ring is a complete intersection. In higher codimension this is no longer true.*

Exercise 8. Show that Example 6.5 (2) proves the second statement in Theorem 6.6.

Theorem 6.7 (Buchsbaum-Eisenbud [11])**.** *Every height 3 Gorenstein ideal, I, can be realized as the ideal generated by the maximal Pfaffians of a $t \times t$ skew symmetric matrix, A, for any odd t. This matrix occurs as the middle matrix in the minimal free resolution of I:*

$$0 \to R(-k) \to \mathbb{F}_2 \xrightarrow{A} \mathbb{F}_1 \to R \to R/I \to 0.$$

Corollary 6.8. *Any height three Gorenstein ideal has an odd number of minimal generators.*

This structure theorem of Buchsbaum and Eisenbud is very powerful, and most results on height 3 Gorenstein ideals rely on it in one way or another. (The main result of [63] is one exception.) An important example is the work of Diesel [16], who gave a careful description of the resolutions that actually occur in the Artinian case, and obtained conclusions about irreducible families of Artinian Gorenstein algebras. Her classification for the Artinian case was extended to reduced sets of points in [26]; see (5) in section 9 below. Another group of papers describes the possible Hilbert functions of Gorenstein algebras under various assumptions – cf. for instance [15], [34].

In the literature, there is a very strong correlation between results about codimension three Gorenstein ideals (or arithmetically Gorenstein subschemes of $\mathbb{P}^n$) and codimension two Cohen-Macaulay ideals (or ACM subschemes of $\mathbb{P}^n$), and the role played by the Hilbert-Burch matrix in the latter case is mimicked by the Buchsbaum-Eisenbud matrix in the latter case. A very recent illustration of this is given by the proof of the Extended Multiplicity Conjecture for these two cases (cf. [61] and [62]; see also [33], [35], [64]).

7. Weak and Strong Lefschetz properties, and "generality" conditions

In this section we will often consider graded Artinian algebras. Hence we will sometimes work over the ring $R := K[x_1, \dots, x_n]$ and sometimes over $P = K[x_0, \dots, x_n]$. Let $A = R/I$ be a graded Artinian algebra. Let L be a *general* linear form. We have, for each $t \in \mathbb{Z}$, a (vector space) homomorphism

$$(R/I)_t \xrightarrow{\times L} (R/I)_{t+1}.$$

Definition 7.1. A has the *Weak Lefschetz property* (WLP) if this homomorphism has maximal rank, for each t.

We can do the same thing for a *general* homogeneous polynomial F of degree d:

$$(R/I)_t \xrightarrow{\times F} (R/I)_{t+d}$$

is again a homomorphism.

Definition 7.2. A has the Strong Lefschetz property (SLP) if this homomorphism has maximal rank, for all d and all t.

It can be shown that WLP and SLP are open conditions on components of the Hilbert scheme parameterizing algebras with given Hilbert function (in vague terms). However, the empty set is also open! In the tutorial we talk a bit about how there can be components of the same Hilbert scheme that behave very differently with respect to WLP, based on the work in [53].

In these notes and in the tutorials, there are three very similar-looking conjectures about "general" situations. We would like to clarify and compare (and contrast) them here.

7.1. **Minimal Resolution Conjecture.** Let Z be a general set of s points in $\mathbb{P}^n$. Then as mentioned above, the Minimal resolution Conjecture says that the minimal free resolution of I_Z has no ghost terms. Let us examine this a little bit further.

First, what is the Hilbert function of Z? First consider the dimensions of the homogeneous components of P:

$$1, \quad \binom{n+1}{1}, \quad \binom{n+2}{2}, \quad \binom{n+3}{3}, \quad \dots$$

Eventually these numbers become greater than s. Since the points are general, they impose the maximum number of conditions on forms of any degree, t. Either this number is s itself (i.e. the points impose independent conditions) or the number is the dimension of the whole component of degree t (so there is *no* form of degree t containing Z). The first degree in which the ideal is not zero is called the *initial degree* of I_Z, and is denoted by α. If $a_1 := \dim(I_Z)_\alpha$, this is the number of minimal generators of least degree, and is forced by the Hilbert function. The Hilbert function of Z has the form

$$1, \quad \binom{n+1}{1}, \quad \binom{n+2}{2}, \quad \dots, \quad \binom{n+\alpha-1}{\alpha-1}, \quad \underbrace{\binom{n+\alpha}{\alpha} - a_1}_{\substack{||\\ s}}, \quad s, \quad s, \quad \dots.$$

and we conclude that $a_1 = \binom{n+\alpha}{\alpha} - s$.

Exercise 9. Verify this Hilbert function computation for an ideal of s general points in $\mathbb{P}^n$.

It is also possible to show that a Hilbert function of this type allows at worst two shifts in each free module of the minimal free resolution. That is, the resolution has the form

$$0 \to \begin{array}{c} P(-\alpha-n+1)^{a_n} \\ \oplus \\ P(-\alpha-n)^{b_n} \end{array} \to \begin{array}{c} P(-\alpha-n+2)^{a_{n-1}} \\ \oplus \\ P(-\alpha-n+1)^{b_{n-1}} \end{array} \to \cdots \to \begin{array}{c} P(-\alpha-1)^{a_2} \\ \oplus \\ P(-\alpha-2)^{b_2} \end{array} \to \begin{array}{c} P(-\alpha)^{a_1} \\ \oplus \\ P(-\alpha-1)^{b_1} \end{array} \to I_Z \to 0.$$

The content of the Minimal Resolution conjecture is that $a_i \cdot b_{i-1}$ should be 0 for all i, and this expected value can be computed from the Hilbert function. For example, b_1 is the number of minimal generators of I_Z in degree $\alpha+1$, and the expected value for b_1 is

$$b_1 = \max\left\{ \left[\binom{n+\alpha+1}{\alpha+1} - s \right] - (n+1) \cdot \left[\binom{n+\alpha}{\alpha} - s \right], 0 \right\}.$$

In a similar way one can compute the expected values of the other a_i and b_i.

Exercise 10. Show that equivalently, we have

$$b_1 = \max\left\{ \binom{n+\alpha}{\alpha+1} - n \cdot \left[\binom{n+\alpha}{\alpha} - s \right], 0 \right\}.$$

As noted above, the Minimal Resolution conjecture is known to be false. The first counterexample was found by Schreyer, but it was all put into a theoretical framework by Eisenbud and Popescu [17]. It consists of 11 general points in $\mathbb{P}^6$. Nevertheless, the Minimal Resolution conjecture is known to hold asymptotically thanks to work of Walter [74] and of Hirschowitz and Simpson [36]. An interesting special case, namely verifying that b_1 has the expected value, is called the Ideal Generation conjecture, and is still open.

7.2. **Artinian Minimal Resolution Conjecture.** This conjecture is the Artinian analog of the Minimal Resolution Conjecture. It can be viewed as an attempt to address the following "philosophical" question: How "general" is the Artinian reduction of a general set of points? The "philosophical" answer to this question is, "Not as general as you might think!" In other words, among all Artinian algebras with a fixed (generic) Hilbert function, over "sufficiently many" variables, there is a Zariski open subset consisting of algebras that are *not* the Artinian reduction of a reduced, finite set of points.

This conjecture will be discussed in the tutorials, but in brief here is the idea. Fix a degree, α, and consider a *generic* K-vector space $V \subset K[x_1, \dots, x_n]$ of dimension $a_1 := \binom{n+\alpha}{\alpha} - s$. Let I be the ideal generated by V and $P_{\alpha+1}$. Of course part of $P_{\alpha+1}$ is already generated by V.

Exercise 11. Check that the expected number of minimal generators in degree $\alpha + 1$ is exactly b_1 as computed above. (Hint: use Exercise 10.)

The other expected graded Betti numbers can be computed as above. The conjecture then states, as above, that there are no ghost terms in the minimal free resolution of I. That is, we have exactly the same predicted minimal free resolution. The difference is that in this Artinian context there are *no* known counter-examples! Even the Artinian analog of the known counter-examples for points (e.g. 11 general points in $\mathbb{P}^6$) fails to pick up ghost terms. Kreuzer has informed us that he has checked this conjecture experimentally for very high values of α and n.

In this context the Artinian version of the Ideal Generation conjecture (i.e. verifying b_1) is known to be proved, and is due to Hochster and Laksov [37]. As noted above, you will revisit this conjecture and perform experiments in the tutorials.

7.3. **Ideals of general forms and the Fröberg Conjecture (and beyond).** In the Artinian Minimal Resolution conjecture we restricted ourselves to forms of the same degree, or at worst of two consecutive degrees. We now allow ourselves complete freedom in choosing the number and degrees of the forms. More precisely, fix $d_1, \dots, d_k$ and consider ideals formed by general choices of forms of these degrees in $R := K[x_1, \dots, x_n]$. We are interested first in the Hilbert function, and then in the minimal free resolution of such an ideal. There is again an expected Hilbert function, and the most compact way of describing it is as follows:

Conjecture 7.3 (Fröberg)**.** *The coordinate ring of an ideal of general forms satisfies the Strong Lefschetz property.*

We leave it as an exercise to see how this translates to a statement about Hilbert functions. This conjecture is known to be true in two variables [19] and three variables [1], and in the case $k = n + 1$ [73], [76]. Of course the result of Hochster and Laksov mentioned above is an important contribution as well. See also [3] and [20].

Less seems to be known about the minimal free resolution. Of course one cannot hope to precisely give the minimal free resolution without first knowing the Hilbert function, but perhaps one can give good properties anyway. The natural first guess is that there should be no ghost terms in the minimal free resolution. This, however, has no chance. For instance, if we start with generators of degrees 4,4 and 8, we see that the two generators of degree 4 have a Koszul first syzygy of degree 8, so unless other generators enter to nullify either the generator of degree 8 or the Koszul syzygy, a ghost term is forced. We will call this a *Koszul ghost term.* The "Thin Resolution Conjecture" of Iarrobino said that a general set of forms generate an ideal for which there are no non-Koszul ghost terms. This was disproved in [54], and together with [55] it was shown how liaison can give a whole class of examples. A simple example (that also shows how Koszul ghost terms arise) is the case of general forms in $K[x_1, x_2, x_3]$ of degrees 4,4,4,8. The minimal free resolution has the form

$$0 \to \begin{pmatrix} R(-10) \\ \oplus \\ R(-11)^2 \end{pmatrix} \to \begin{pmatrix} R(-8)^3 \\ \oplus \\ R(-9)^2 \\ \oplus \\ R(-10) \end{pmatrix} \to \begin{pmatrix} R(-4)^3 \\ \oplus \\ R(-8) \end{pmatrix} \to R \to R/I \to 0$$

We see the Koszul ghost term $R(-8)$ as predicted, but also the non-Koszul ghost term $R(-10)$. In [54] the minimal free resolution for four general forms of arbitrary degree in 3 variables was completely solved, using liaison and the work of Diesel [16].

8. Liaison Theory

In this section we offer a very sketchy introduction to liaison theory. This is a very active area recently. For more details we refer the reader to the book [51], the monograph [40] and to the expository notes [59]. The purpose here is really to give the student an overview of the main definitions and questions and a few of the results, but including almost nothing of the tools and methods used in obtaining the main results. The student is encouraged to follow the references to get the details.

8.1. **Overview.** The idea governing liaison theory, which goes back more than a century, is that if two closed projective subschemes (or two varieties, classically), V_1 and V_2, are such that

(a) they have the same dimension and their saturated ideals are unmixed (i.e. no embedded or isolated components),
(b) they have no common component, and
(c) their union is a complete intersection, X (i.e. $I_{V_1} \cap I_{V_2} = I_X$)

then a lot of information is passed from one to the other.

Definition 8.1. If the three assumptions above are satisfied, we say that V_1 and V_2 are *geometrically linked*. For emphasis, we sometimes say that they are *geometrically CI-linked*. *Geometric CI-liaison* is the equivalence relation generated by geometric CI-links.

Note that geometric CI-links are not reflexive or transitive in general, so it is necessary to speak of the equivalence relation *generated* by such links. For a zero-dimensional example, in Figure 1, the complete intersection, X consists of the intersection in $\mathbb{P}^2$ of the six "vertical" lines and the four "horizontal" lines, giving 24 points in all. The solid dots, V_1, are thus linked via this complete intersection to the open dots, V_2. One of our first important results is that in $\mathbb{P}^2$, any zero-dimensional scheme is in the same CI-liaison class as any other zero-dimensional scheme. This is not true (with CI-liaison) in higher codimension.

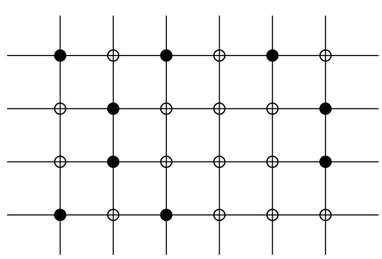


FIGURE 1. Geometric link of points

Figure 2 gives a one-dimensional example. Here the complete intersection again has codimension two, and this time is the intersection of the surface of degree 2 (a union of planes) and the surface of degree 1. The resulting curve, X, is the union of lines V_1 and V_2.

Under this situation, a lot of information can be passed from one to the other, taking into account the complete intersection, X. Typically, one starts with V_1, finds a complete

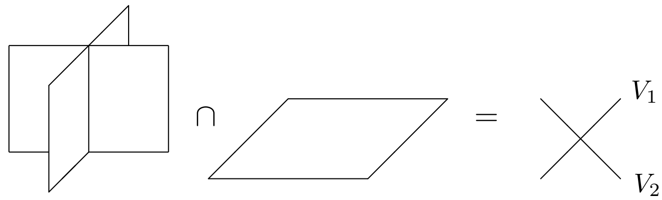


FIGURE 2. Geometric ink of curves

intersection containing it, and looks at the "residual" V_2. On the computer, the best way is via a program such as `GenResSeq(I,L)` that you wrote in the tutorial.

The hope, classically, was that one could start with an arbitrary V_1, even general, and always "link" down to something simpler, in a series of steps that one understands. In this way one could get information about V_1 from the simpler linked varieties. This works well for codimension two ACM subschemes, but not otherwise in general. This is reflected in the Lazarsfeld-Rao property, discussed below, and was first pointed out by Joe Harris.

What kind of information is passed from V_1 to V_2? Here are some examples.

(1) We have the *degree formula* $\deg V_1 + \deg V_2 = \deg X$. This is obvious since V_1 and V_2 have no common components, but it is far less obvious in the more general context described below.
(2) Suppose that V_1 and V_2 are curves in $\mathbb{P}^n$ whose union, X, is a complete intersection of type $(a_1, \dots, a_{n-1})$. Let $a := a_1 + \cdots + a_{n-1}$. Let g_1 and g_2 denote the arithmetic genera of V_1 and V_2, respectively. Then we have the *genus formula*
$$g_1 - g_2 = \frac{1}{2}(a - n - 1)(\deg V_1 - \deg V_2).$$
Note that a is the twist of the last free module in the minimal free resolution of I_X.
(3) The property of being ACM is preserved (and an even stronger statement involving the cohomology of V_1 and V_2 is true – see below).
(4) In the ACM case, the Hilbert function of V_2 can be computed from that of V_1. In the non-ACM case there is still a formula, but it relies also on the cohomology.

It turns out that the theory is rather complete for codimension two liaison. Now, recall also from Theorem 6.6 that in codimension two, the arithmetically Gorenstein subschemes and the complete intersections coincide. To extend to higher codimension, then, it is not obvious a priori whether our links should continue to be complete intersections, or arithmetically Gorenstein subschemes. Both directions have proven to be fruitful, although the latter is a more recent and very active area of research.

8.2. **Algebraic Linkage.** Notice that deg $V_1 +$ deg $V_2 =$ deg X in both examples above. The simplest example of two curves that are geometrically directly linked is the union of two lines, given as the intersection of a pair of planes with another plane (see Figure 2). The problem comes when we try to extend this notion to the case where the curves may have common components. For example, if the second surface (the plane) contains the line of intersection of the two planes comprising the first surface, this is still a perfectly good complete intersection (see Figure 3).

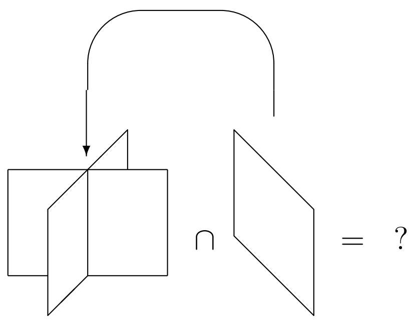


FIGURE 3. Algebraic Link

The only natural way to interpret this would be to say that the line of intersection is linked to itself (since the complete intersection still has degree two), but using only unions of course we do not have the equality in the definition.

The solution is to use ideal quotients. Furthermore, in order to be as general possible, we will allow X to be simply arithmetically Gorenstein instead of insisting that it be a complete intersection. Indeed, our goal will be to remove the assumption (b) at the beginning of this section, and to weaken (c) as much as possible.

Definition 8.2. If V_1 and V_2 are subschemes of $\mathbb{P}^n$ such that

(a) they have the same dimension and their saturated ideals are unmixed (i.e. no embedded or isolated components),
(b) they have no common component, and
(c) their union is an arithmetically Gorenstein scheme, X (i.e. $I_{V_1} \cap I_{V_2} = I_X$)

then we say that V_1 and V_2 are *geometrically G-linked*. *Geometric G-liaison* is the equivalence relation generated by geometric G-links.

Definition 8.3. Two subschemes V_1 and V_2 are *algebraically G-linked* if there is a Gorenstein ideal $I_X \subset I_{V_1} \cap I_{V_2}$ such that

$$\begin{aligned} I_X : I_{V_1} &= I_{V_2} \\ I_X : I_{V_2} &= I_{V_1}. \end{aligned}$$

If X is a complete intersection then we say that V_1 and V_2 are *algebraically CI-linked*. *Algebraic G-liaison* (resp. *algebraic CI-liaison*) is the equivalence relation generated by algebraic G-links (resp. algebraic CI-links). If V_1 is linked to V_2 we will write $V_1 \sim V_2$, or sometimes $V_1 \overset{X}{\sim} V_2$ if we want to emphasize the specific Gorenstein scheme that performs the link.

Here are some first results about linkage.

Theorem 8.4. (1) *If V_1 and V_2 are geometrically linked then they are algebraically linked.*

(2) *If V_1 and V_2 are algebraically linked then* automatically *they are unmixed. More precisely, the process of performing the ideal quotient $I_X : I_{V_1}$ gives a residual that is unmixed. In fact, the double ideal quotient $I_X : [I_X : I_{V_1}]$ gives the top dimensional part of V_1 so the links lose the information of the lower dimensional components that you started with. (On CoCoA this is an effective algorithm for computing the top dimensional part of an ideal.)*

(3) *If V_1 is unmixed and $I_X \subset I_{V_1}$ is Gorenstein, then $I_{V_2} := I_X : I_{V_1}$ is algebraically linked to I_{V_1}. That is, the second equality in the definition automatically holds.*

(4) *The degree formula (1) continues to hold under algebraic linkage.*

(5) *If V_1 and V_2 are curves then the genus formula (2) continues to hold for algebraic CI-linkage. Furthermore, if I_X is Gorenstein with minimal free resolution ending*

$$0 \to R(-a) \to \dots$$

(where a now is no longer necessarily the sum of the degrees of the generators of I_X) then the genus formula continues to hold for algebraic G-linkage.

A result about CI-liaison that was known classically, and whose proof is not so hard to write, is the following. What is really interesting is that there is an analog in G-liaison (Theorem 8.6), but it is a deep and very recent result.

Theorem 8.5. *Any two complete intersections of the same codimension in $\mathbb{P}^n$ are in the same CI-liaison class.*

Proof. Although this has been known for a long time, the only written proof of which I am aware is in the thesis of Phil Schwartau [72]. It is based on the following lemma: If $I_{X_1} = (F_1, \dots, F_{d-1}, F)$, $I_{X_2} = (F_1, \dots, F_{d-1}, G)$ and $I_X = (F_1, \dots, F_{d-1}, FG)$ then X_1 is directly linked to X_2 by the complete intersection X. Verifying this and finding the rest of the proof is left as an exercise. $\square$

Theorem 8.6 ([12])**.** *Any two arithmetically Gorenstein subschemes of the same codimension in $\mathbb{P}^n$ are in the same G-liaison class.*

Because of Theorem 8.5, it makes sense to speak of the liaison class of a complete intersection, and many papers have been written to try to determine when an ideal (or subscheme) is in this class. Of course the most complete answer is in codimension two, as we will see.

Definition 8.7. A subscheme $V \subset \mathbb{P}^n$ is *licci* if it is in the CI-**li**nkage **c**lass of a **c**omplete **i**ntersection. V is *glicci* if it is in the **G**orenstein-**li**nkage **c**lass of a **c**omplete **i**ntersection.

8.3. **Some basic questions (and a few answers) about liaison.** In this subsection we point out some natural questions that arise from the definitions and first results that we have given above. In some cases we provide some comments as well, and in subsequent subsections we will discuss other answers.

(1) In view of Theorem 8.4 (1), it is conceivable that geometric linkage (CI or G) generates a *different* equivalence relation than algebraic linkage (CI or G). This is known to be true for CI-liaison, but still open for G-liaison.
(2) Are G-liaison and CI-liaison the same equivalence relation? In other words, it is obvious that if two subschemes are CI-linked in a finite number of steps then they are G-linked in a finite number of steps. But is the converse true? See Example 8.8.
(3) One might ask why we have to stop with X arithmetically Gorenstein, and cannot in fact allow X to be arithmetically Cohen-Macaulay. In the case of points in $\mathbb{P}^2$ (or indeed, in any $\mathbb{P}^n$) it is clear that there is nothing to study, since any zero-dimensional scheme is ACM, so the same is true of the union.

    More interestingly, it was shown by Charles Walter [75] that geometric CM-liaison is a trivial equivalence class. (Sometimes showing that something is trivial is a deep result!!) See also Exercise 12 below.
(4) What are necessary and sufficient conditions for two subschemes to be in the same liaison class, CI or G? This is known in codimension two but is open in general.
(5) Do the liaison classes themselves have any common structure?
(6) Are there any interesting applications of liaison theory?
(7) It was noted above that (direct) linkage is rarely reflexive. That is, it is almost never true that for a given scheme V there is a complete intersection (or an arithmetically Gorenstein scheme), X, for which $I_V : I_X = I_V$. Of course under geometric links this has no chance, but even under algebraic links it is not possible. As a trivial example, if V is a set of 7 points in $\mathbb{P}^2$ in general position then it lies on no conics. But in order for $I_X : I_V = I_V$, we need V to be linked to itself by a complete intersection of degree 14, and the prime factorization of 14 is (2)(7). Hence this is impossible.

    Subschemes V of $\mathbb{P}^n$ for which $I_X : I_V = I_V$ for some arithemtically Gorenstein scheme X (possibly a complete intersection) are called *self-linked*. These are very interesting. We refer the student to the paper [70] of Rao for many results on this subject.

Exercise 12. Find an ACM curve, X, and a curve C_1, with $I_X \subset I_{C_1}$, such that if $I_{C_2} := I_X : I_{C_1}$ then

$$I_X : I_{C_2} \neq I_{C_1}, \quad \text{and} \quad \deg C_1 + \deg C_2 \neq \deg X.$$

Example 8.8. Let C_1 be a rational normal curve in $\mathbb{P}^4$ and let C_2 be a line in $\mathbb{P}^4$. Then C_1 and C_2 are G-linked in finitely many steps (in fact it can be done in three or fewer steps), but they are not CI-linked in any number of steps. (This can be shown using Theorem 2.5 of [44], for instance.)

8.4. **ACM subschemes.** We have focused quite a bit in these notes on the general topic of ACM subschemes of $\mathbb{P}^n$, and indeed they are the most basic in many ways (as we have seen). It makes sense to turn to them also in the context of liaison theory. The most complete answer, as one might expect, comes in codimension two. The following was known classically, thanks to work of Apery [2] and Gaeta [21, 22], and was placed in a modern setting by Peskine and Szpiro [67].

Theorem 8.9 (Apery, Gaeta, Peskine-Szpiro). *The codimension two ACM subschemes of* $\mathbb{P}^n$ form *a liaison class.*

We mentioned already that the ACM property is preserved, so the new "half" of this theorem says that any two ACM codimension two subschemes are in the same liaison class (and since it is codimension two, we do not have to specify CI or G). That is, a codimension two ACM subscheme of $\mathbb{P}^n$ is *licci*. We saw in Example 8.8 that in codimension three (or more) this is no longer true: there are ACM subschemes that are not licci. However, we have the following conjecture:

Conjecture 8.10. *Any codimension 3 ACM subscheme of* $\mathbb{P}^n$ *is* glicci.

This was first asked as a question in [40], Question 1.6. In fact, the question was asked for ACM subschemes of any codimension, but to hedge our bets a little here, we conjecture it only in codimension three.

Note that Theorem 8.9 implies, in particular, that any zero-dimensional scheme in $\mathbb{P}^2$ is licci, so they are all in the same liaison class.

Theorem 8.9 is not hard to prove. For instance, it follows very quickly from some standard methods (the so-called *mapping cone construction*) that if V_1 is ACM of codimension two and $I_X \subset I_{V_1}$ is obtained using minimal generators of I_{V_1} (which is always possible), then $I_{V_2} := I_X : I_{V_1}$ is the saturated ideal of the residual ACM subscheme, V_2, and I_{V_2} has one fewer minimal generator than does I_{V_1}. This observation alone is enough to prove that V_2 is licci, and hence to prove the theorem. But Gaeta gave a much deeper analysis of the minimal generators and the Hilbert-Burch matrices involved.

8.5. **Non-ACM curves: necessary and sufficient conditions for linkage.** For the non-ACM case, we will restrict to curves, for simplicity. But it should be noted that

Rao has solved the problem in the general case of codimension two [69] locally Cohen-Macaulay subschemes. It was extended to the non-locally Cohen-Macaulay situation by Nagel [65] and Nollet [66].

In the interest of keeping these notes as simple as possible, we will also restrict ourselves to *even* liaison for the non-ACM case.

Definition 8.11. Two subschemes $V, V' \subset \mathbb{P}^n$ are *evenly linked* if there is a sequence of links

$$V \sim V_1 \sim \cdots \sim V_k \sim V'$$

where k is odd (so there is an even number of links). The set of all subschemes that can be obtained in this way is called the *even liaison class of* V, and the equivalence relation so generated is called *even liaison.*

The case of curves in $\mathbb{P}^3$ was solved by Rao in a separate paper, [68]. Recall that above we defined the *deficiency module*, also known as the *Hartshorne-Rao module*, of a closed subscheme of $\mathbb{P}^n$. More precisely, if the closed subscheme is a curve C (in our setting), we have

$$M(C) := \bigoplus_{t \in \mathbb{Z}} H^1(\mathbb{P}^n, \mathcal{I}_C(t)).$$

We have seen that C is ACM if and only if $M(C) = 0$. It is also true that if I_C is unmixed (as we are assuming), then $M(C)$ has finite length. Furthermore, if you are not comfortable with sheaf cohomology, there is an isomorphism between the K-dual of $M(C)$ and

$$\mathrm{Ext}^n_P(P/I_C, P)(-n-1).$$

This also has the advantage of being much easier to compute on CoCoA (although currently the Ext function has a bug, which the CoCoA team promises will be fixed soon).

Example 8.12. Let C be a set of two skew lines in $\mathbb{P}^3$. We can compute $M(C)$ either through the sheaf definition or via Ext. In the former case, the short exact sequence of sheaves

$$0 \to \mathcal{I}_C(t) \to \mathcal{O}_{\mathbb{P}^3}(t) \to \mathcal{O}_C(t) \to 0$$

leads to the long exact sequence in cohomology

$$0 \to H^0(\mathbb{P}^n, \mathcal{I}_C(t)) \to H^0(\mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(\mathbb{P}^n, \mathcal{O}_C(t)) \to H^1(\mathbb{P}^n, \mathcal{I}_C(t)) \to 0,$$

which in turn translates (via isomorphism) to

$$0 \to (I_C)_t \to P_t \to H^0(\mathbb{P}^n, \mathcal{O}_C(t)) \to M(C)_t \to 0.$$

The third term is easy to compute since C is the union of two $\mathbb{P}^1$'s. We see that $M(C) \cong K$, and the unique non-zero component occurs in degree 0.

From the Ext point of view, it is easy to compute using the minimal free resolution

$$0 \to P(-4) \to P(-3)^4 \to P(-2)^4 \to P \to P/I_C \to 0.$$

The relevance of this module to liaison was first noted by Gaeta, who observed that the vector space dimensions of its components are preserved up to re-ordering and shifting. But the first modern proof that this module is important to liaison is due to Hartshorne, who proved the following.

Theorem 8.13 (Hartshorne)**.** *Let C_1 and C_2 be evenly linked curves in $\mathbb{P}^n$. Then $M(C_1)$ is isomorphic to some shift of $M(C_2)$.*

There is an analog for odd liaison which also involves the dual of $M(C)$. Also, it is not hard to extend this theorem (including the case of an odd number of links) to subschemes of any dimension [50].

The amazing result, due to Rao, is that the converse of Theorem 8.13 is true for curves in $\mathbb{P}^3$.

Theorem 8.14 ([68])**.** *Let $C_1, C_2 \subset \mathbb{P}^3$. If $M(C_1)$ is isomorphic to some shift of $M(C_2)$ then C_1 is evenly linked to C_2.*

Again, there is a version for odd liaison. There is also a theorem for codimension two, but it is phrased in terms of stable equivalence classes of vector bundles rather than in terms of the cohomology modules.

It is known that Theorem 8.14 is not true for curves in higher projective space, at least under complete intersection links. For instance, in Example 8.8 both curves have trivial deficiency module. An open question is the following, which we will state as a conjecture:

Conjecture 8.15. *If $M(C_1) \cong M(C_2)(\delta)$ for some $\delta \in \mathbb{Z}$ for curves $C_1, C_2 \subset \mathbb{P}^n$ (or at least $\mathbb{P}^4$, say) the C_1 and C_2 are evenly G-linked.*

Exercise 13. The following exercises can be done on CoCoA, and test both your understanding of the material and your abilities with CoCoA.

(1) Use liaison to find smooth curves $C_1, C_2 \subset \mathbb{P}^3$, both of degree 6 and genus 3, such that C_1 is ACM and CF_2 is not ACM.
(2) Construct an ACM curve of degree 2005 in $\mathbb{P}^3$. Construct a non-degenerate, non-ACM curve of degree 2005 in $\mathbb{P}^4$.
(3) (a) Let Z_1 be a set of three non-collinear points in $\mathbb{P}^3$ and X_1 a complete intersection of three quadrics containing Z_1 (i.e. $I_{X_1} = (Q_1, Q_2, Q_3) \subset I_{Z_1}$). Let Y_1 be the residual (i.e. linked) set of points. Find the minimal free resolution of I_{Y_1}.
    (b) Let Y_2 be a set of eight points on a twisted cubic curve in $\mathbb{P}^3$. Find a way to construct such a set of points. Find the minimal free resolution of I_{Y_2}.
    (c) Let Z_3 be a set of 13 general points on a smooth quadric surface in $\mathbb{P}^3$. Find a way to construct such a set of points in $\mathbb{P}^3$. Let X_3 be a sufficiently general

complete intersection of type $(3,3,3)$ containing Z_3, and let Y_3 be the residual set of points. Find the minimal free resolution of I_{Y_3}.

(d) What do you notice that the above mimimal free resolutions have in common?

Inspired by the above examples, and extending the notion of a complete intersection, we have the following.

Definition 8.16. An *almost complete intersection (ACI)* is an ACM subscheme of $\mathbb{P}^n$ whose number of minimal generators is one more than the codimension. An Artinian ideal, $I \subset K[x_1, \dots, x_n]$, is an almost complete intersection if it has $n+1$ minimal generators.

This provides a rather useful way of obtaining arithmetically Gorenstein subschemes of projective space, via the following theorem. The proof is via the mapping cone procedure mentioned above, and we omit the details (but refer the reader to [51]).

Theorem 8.17. *If V is an ACI and if X is a complete intersection such that $I_X \subset I_V$ and the minimal generators of I_X are all minimal generators of I_V, then the residual scheme V' defined by $I_X : I_V =: I_{V'}$ is arithmetically Gorenstein.*

This was used to great advantage in [54], where ideals of $n+1$ general forms in n variables were studied. This was especially fruitful when $n = 3$, thanks to the fact that Diesel [16] had classified the possible minimal free resolutions of Gorenstein ideals. This reflects a successful application of the classical philosophy described at the beginning of this section: we use liaison to transfer a problem to a situation that we understand, and pull the information back to the study of the ideals in which we are actually interested.

8.6. **The structure of an even liasion class.** Whenever you have an equivalence relation on a set (e.g. the integers modulo n) it is of interest to know (a) how many equivalence classes there are, (b) what does the set of equivalences classes look like, (c) necessary and sufficient conditions for two elements to be in the same class, and (d) whether any one class has any particular structure (and in particular, if all classes have the same structure).

In this section we describe what is known about this latter question for the equivalence relation of even liaison. As with many other questions about liaison, the only known result is in codimension two, and there we have a fairly complete picture. As usual, we refer to [51] for the details, and we just give the idea.

We have seen that given a curve C in $\mathbb{P}^3$, the Hartshorne-Rao module $M(C)$ is an invariant, up to shift, for the even liaison class of C. But what shifts occur? We assume now that C is not ACM, for the sake of discussion of the shifts of the module, although basic double linkage works perfectly well for ACM curves as well. A very simple construction called *basic double linkage*, introduced by Lazarsfeld and Rao [46] has the effect of starting with C and producing a new curve C_1 with the property that $M(C_1) \cong M(C)(-d)$ for some (any) $d \geq 1$. (Note that $M(C)(-d)$ is a *rightward* shift of the module by d places.) The ideal of I_{C_1} is given by

$$I_{C_1} = G \cdot I_C + (F),$$

where $F \in I_C$ is a homogeneous polynomial of any degree, and $G \in P_d$ is a homogeneous polynomial of degree d, such that (F, G) is a regular sequence. Geometrically, C_1 is obtained by starting with C, choosing a surface F containing C, and adding to C the hypersurface section of F cut out by G. The process is called "basic double linkage" because, as the name suggests, C_1 is linked to C in two steps. (It is not hard to see this, and is left as an exercise.)

What about the other direction? It is not hard to show (see e.g. [49]) that for d sufficiently large, $M := M(C)(d)$ (a *leftward* shift) is *not* the Hartshorne-Rao of any curve. (This was generalized to any dimension in [8].) Hence clearly there is a largest d for which $M(C)(d)$ actually occurs as the Hartshorne-Rao module of some curve. One might hope that the set S_1 of curves in the even liaison class of C for which the module occurs in this leftmost shift is special in some way.

One can also consider the set S_2 of curves in the even liaison class of C that have the smallest possible degree. Clearly if C is not ACM then there is no chance of finding a curve of degree 1. Do all even liaison classes contain a curve of degree 2? In any case, the degree is clearly bounded below (and not above, by the discussion above). Similarly, one can consider the set S_3 of curves in the even liaison class of C that have the smallest arithmetic genus.

What is surprising is that S_1, S_2 and S_3 coincide! This is a consequence (among many!) of the *Lazarsfeld-Rao Property*. More precisely, let $\mathcal{L}$ be an even liaison class of curves in $\mathbb{P}^3$. We partition $\mathcal{L} = \bigcup_{t \geq 0} \mathcal{L}^t$ according to the shifts of the Hartshorne-Rao module, where $\mathcal{L}^0 = S_1$ (described above). The Lazarsfeld-Rao Property says that

- Given any two elements, C_0 and C_0', of $\mathcal{L}^0$, there is a flat deformation from C_0 to C_0' through curves all in $\mathcal{L}^0$. The elements of $\mathcal{L}^0$ ar called the *minimal elements* of the even liaison class $\mathcal{L}$.
- Given any minimal element $C_0 \in \mathcal{L}^0$ and any $C \in \mathcal{L}^h$ (with $h \geq 1$), there is a sequence of basic double linked curves $C_0, C_1, \ldots, C_k$ $(k \leq h)$, such that there exists a flat deformation from C_k to C all through curves in $\mathcal{L}^h$. In fact this sequence of basic double links can be assumed to each use a linear form for the polynomial G (see the definition of basic double linkage above), in which case we have $k = h$.

This property was shown by Lazarsfeld and Rao [46] to hold for the even liaison class of a "general" curve of large degree in $\mathbb{P}^3$. (But a caveat is that the even liaison class of a "general curve" is actually rather special.) The property was proposed to hold in much greater generality, and important tools were developed and first cases proved, by Bolondi and Migliore in [8]. The property was shown to hold for all even liaison classes of curves in $\mathbb{P}^3$ by Martin-Deschamps and Perrin [48], who in fact showed a great deal more. At about the same time it was shown by Ballico, Bolondi and Migliore [4] to hold for all locally Cohen-Macaulay codimension two even liaison classes in $\mathbb{P}^n$. This was extended in different directions by Bolondi and Migliore [10], Nagel [65] and Nollet [66].

Observe that knowing that the Lazarsfeld-Rao Property holds in effect tells you that if you "know" one minimal element of the even liaison class, in many ways you know the whole class. For instance, you know all the possible degrees and genera that occur in the class! In [9] this idea was used to show that all arithmetically Buchsbaum curves specialize to stick figures, a special case of the more general Zeuthen problem (that was later shown to fail to hold in full generality by Hartshorne [32]).

8.7. **Higher codimension and divisors.** What about all of these questions in higher codimension? As you will see in Problems 10 and 11 in Section 9, the problem of finding a structure like the Lazarsfeld-Rao Property is wide open, for both CI and Gorenstein liaison. It may well be that there is nothing so nice to be found. As for necessary and sufficient conditions for linkage, we gave some conjectures above, but again the big questions are open.

It should be noted, though, that a broad picture has emerged of Gorenstein liaison in any codimension as a theory of divisors on ACM subschemes. See [51] for an exposition; another expository paper is [52]. It is based in large part on the following result, which is called *basic double G-linkage*. As with basic double linkage, it is very useful in liaison theory, and also as with basic double linkage it is also very useful beyond liaison theory.

Theorem 8.18 ([40]). *Let $I \subset J$ be saturated homogeneous ideals of P such that* $\operatorname{codim} I + 1 = \operatorname{codim} J$. *Let $G \in P$ be a form of degree d such that $I : G = I$. Let $J_1 := I + G \cdot J$. Then*

(a) $\deg J_1 = d \cdot (\deg I) + \deg J$ *(where* deg *refers to the degree of the scheme defined by the ideal).*
(b) *If I is ACM and smooth then J_1 is G-linked to J in two steps. (We assume "smooth" here for simplicity; a much more general statement is in fact true.)*
(c) *We have a short exact sequence*

$$0 \to I(-d) \to J(-d) \oplus I \to J_1 \to 0,$$

where the first map is given by $F \mapsto (F, GF)$ and the second map is given by $(A, B) \mapsto GA - B$.

Note that the above also works if J is Artinian and I defines a zero-dimensional scheme. If $I = I_S$ and $J = I_C$, $J_1 = I_{C_1}$ (say), then C and C_1 are divisors on S, and C_1 is obtained from C by adding the hypersurface section cut out by G. Compare this with the geometric description of basic double linkage given above. The following was also shown in [40]:

Theorem 8.19. *Let S be a smooth (again, this can be generalized) ACM subscheme of $\mathbb{P}^n$ and let C be a divisor on S. Then any effective divisor $C' \in |C+tH|$ is evenly G-linked to C, where H is the class of a hyperplane section and t is any integer, positive or negative.*

Many partial results along the lines of the conjectures given above have been shown using this result.

9. Open Problems

In this section we highlight some open problems associated with the material in these notes. The solution of some would attract more attention than others, but all are of interest.

(1) We know the possible Hilbert functions for sets of points in projective space, $\mathbb{P}^n$. What are the possible Hilbert functions for sets of points in $\mathbb{P}^n$, $n \geq 3$, with UPP?
(2) What are the possible Hilbert functions of Artinian Gorenstein algebras of codimension $c \geq 4$? What are the possible minimal free resolutions for Artinian Gorenstein algebras of codimension $c \geq 4$? (This problem is probably not tractable.)
(3) As a special case of great interest, it is known that in codimension 3 all Gorenstein algebras have Artinian reduction with unimodal Hilbert function, and it is known that in codimension ≥ 5 this is not true, but it is open whether it is true or not in codimension 4. In fact, it is not known if there is any reduced set of points that is arithmetically Gorenstein, but whose h-vector is not unimodal.
(4) Do all Gorenstein algebras in codimension three have Artinian reduction with WLP? A special case of this was shown in [29]: every codimension three Artinian complete intersection has WLP. To prove this it was necessary to invoke some unexpected machinery, in particular the Grauert-Mülich theorem on rank two vector bundles on $\mathbb{P}^2$.
(5) Is it true that for any Hilbert function that exists for an Artinian Gorenstein algebra, there is a reduced arithmetically Gorenstein set of points whose Artinian reduction has this Hilbert function? The best result in this direction is that in [58], where it is shown in the affirmative for any so-called *SI-sequence*. In codimension 3 it was shown in [26] that even for every set of graded Betti numbers that exists for an Artinian Gorenstein algebra there exists a reduced, arithmetically Gorenstein set of points whose Artinian reduction has this set of graded Betti numbers.
(6) Find other constructions of Gorenstein (or level) algebras besides those given above. One was given in [7] and [6].
(7) Almost all of the above questions can be asked also for *level algebras* (see Remark 6.2). But here even in three variables very little is known. The problem is that we

are missing a structure theorem analogous to the Buchsbaum-Eisenbud theorem mentioned above. We focus on three questions for the case of level algebras in three variables. All of these are considered carefully in [23], but we are far from a complete solution.

(a) What Hilbert functions can occur for an Artinian level algebra in three variables? Recall that this is known for the Gorenstein case. Some partial results can also be found in several recent papers of Zanello.
(b) Is it true that every Hilbert function that occurs for an Artinian level algebra in three variables lifts to a reduced set of points (with level Artinian reduction, of course)?
(c) Does every Artinian level algebra in three variables have the Weak Lefschetz Property? A weaker question is whether every Artinian level algebra in three variables has unimodal Hilbert function. [Note: after our course ended, a paper of Zanello appeared [77] which gave examples of level Artinian algebras with non-unimodal Hilbert functions, and with unimodal Hilbert functions but failing to have the Weak Lefschetz Property.]

(8) The questions and conjectures in Section 7 are of great interest (and possibly of great difficulty).
(9) Similarly, the questions and conjectures in Section 8 are of great interest and are being actively studied, especially Conjecture 8.10 and Conjecture 8.15.
(10) Is the Lazarsfeld-Rao Property (or some analog) true for even CI-liaison classes in higher codimension?
(11) In higher codimension the Lazarsfeld-Rao property is known to fail for Gorenstein liaison (cf. [51]). Find a good conjecture for the structure of an even Gorenstein liaison class in higher codimension. You get more points if your proposed property specializes to the known Lazarsfeld-Rao Property in codimension two, and even more points if you prove your conjecture!

Exercise 14. Write programs to test some of these questions, and make conjectures if possible.

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