

# Combinatorics of Syzygies

Toric Resolutions

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**Exercise 1.** Let  $A \in \mathbb{N}^{d \times n}$  where we assume that the first row of  $A$  is  $(1, 1, \dots, 1)$ . Using the `Toric` command compute a Gröbner basis and a minimal generating set for the toric ideal  $I_A$  for various matrices  $A$ . Try  $2 \times 4$ ,  $2 \times 5$ ,  $3 \times 5$  and  $3 \times 6$  matrices.

Next develop a `CoCoA` function called `ToricFiber` that takes as an input a toric ideal  $I_A$  and a monomial  $x^u$  and computes the fiber that  $x^u$  belongs to, i.e.  $\{x^v : Av = Au\}$ . (Hint: let  $\mathcal{G}$  be a Gröbner basis  $I_A$ . Given  $x^u$  you can compute the unique normal form of  $x^u$  with respect to  $\mathcal{G}$ .)

**Exercise 2.** Generate random  $d \times (d + 2)$  matrices and their corresponding codimension two toric ideals  $I_A$ . For each instance compute the minimal free resolution of  $S/I_A$ , and compute the fibers of the multidegrees the syzygies belong to. Catalog your results. Any patterns?

**Exercise 3.** Generic toric ideals are, in principal, abundant, but it is still not so easy to generate them. Let's look at the case where  $A = [a_1, a_2, a_3, a_4]$  where  $a_1 < a_2 < a_3 < a_4$ . The first example appears when  $a_1 + a_2 + a_3 + a_4 = 100$ . Make an exhaustive search for the cases when  $101 \leq a_1 + a_2 + a_3 + a_4 \leq 150$ . To make the computations more efficient discard the cases when  $\gcd(a_1, \dots, a_4) \neq 1$  and when one  $a_i$  divides another  $a_j$ . For each instance you find compute and draw the Scarf complex of a reverse lexicographic initial ideal.

**Exercise 4.** Repeat Exercise 3 with randomly chosen  $a_1, a_2, a_3, a_4$ . How much can you push your computations? How many minimal generators do you get?

**Exercise 5.** Prove the following two statements:

- a) A generic monomial ideal  $I$  is Cohen-Macaulay if and only if  $\Delta_I$  is pure.
- b) A generic monomial ideal  $I$  has the *chain property*: if  $P$  is an associated prime of  $I$  then there is a chain of associated primes  $P = P_0, P_1, P_2, \dots, P_k = Q$  where  $Q$  is a minimal prime of  $I$  and  $\dim(P_{i+1}) = \dim(P_i) + 1$  for all  $i = 0, \dots, k - 1$ .

**Exercise 6.** Compute the Scarf complex of all the generic deformations of  $I^3$  where  $I = \langle x, y, z \rangle$ . Also develop a *CoCoA* function *MakeGeneric* that will deform a given monomial ideal to a generic monomial ideal. Can you modify your code in Exercise 4 to compute the irredundant irreducible decomposition of a (not necessarily generic) monomial ideal  $I$ ?

**Exercise 7.** In Lecture 4 we saw that a generic monomial ideal  $I$  in  $k[x, y, z, w]$  where every pair of generators form an edge of  $\Delta_I$  can have at most 12 minimal generators. Can you give two such extremal monomial ideals with non-isomorphic Scarf complexes? A more difficult question: can you classify all Scarf complexes of these extremal monomial ideals?