Combinatorics of Syzygies

Scarf complexes
Tuesday, May 24, 4:00pm
Anna Bigatti and Serkan Hoşten

Exercise 1. Develop a `CoCoA` function called `ScarfComplex` that takes as an input a generic monomial ideal I and lists the facets of its Scarf complex Δ_I. Try to write an efficient code that uses the minimal free resolution of I. Generate three random generic monomial ideals in three variables with ten generators and draw their Scarf complexes. Also generate two random artinian monomial ideals in four variables with eight generators and draw their Scarf complexes.

Exercise 2. Let $u = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ such that $0 < u_1 < u_2 < \cdots < u_n$. Let $I(u) = \langle x_1^{u_{\pi(1)}} x_2^{u_{\pi(2)}} \cdots x_n^{u_{\pi(n)}} : \pi \in S_n \rangle$ where S_n is the symmetric group on n elements. Compute the minimal free resolution of $I(u)$ for $n = 2, 3, 4$ and identify this as a cellular resolution.

Exercise 3. Let I be a monomial ideal and let D be the biggest exponent of any variable in the minimal generators of I. Define $I^* = I + \langle x_1^{D+1}, \ldots, x_n^{D+1} \rangle$. Compute the Scarf complex of I and I^* for a few random examples of generic monomial ideals. State and prove a theorem.

Exercise 4. Let $I = \langle m_1, m_2, \ldots, m_r \rangle$ be a monomial ideal with minimal generators $m_1, \ldots, m_r$. Suppose $m_1 = x_i^{u_i} m'$ for some i where $m' \neq 1$ and m' is not divisible by x_i. Prove that $I = \langle x_i^{u_i}, m_2, \ldots, m_r \rangle \cap \langle m', m_2, \ldots, m_r \rangle$. Using this recursively prove also that I has an *irredundant irreducible decomposition*, i.e., $I = \cap Q_i$ where Q_i is generated by powers of some variables and $Q_i \not\subseteq Q_j$ when $i \neq j$. Write a `CoCoA` function called `IrrepDecomp` that computes the irredundant irreducible decomposition of a given monomial ideal.

Exercise 5. Compare the irredundant irreducible decompositions of I and

I^* for a generic monomial ideals as in Exercise 3. Can you read off these decompositions from Δ_I and Δ_{I^*}? Based on your observations write a code that will compute the irredundant irreducible decomposition of a generic monomial ideal.

Exercise 6. Prove the following two statements:
a) A generic monomial ideal I is Cohen-Macaulay if and only if Δ_I is pure.
b) A generic monomial ideal I has the *chain property*: if P is an associated prime of I then there is a chain of associated primes $P = P_0,\ P_1,\ P_2, \ldots, P_k = Q$ where Q is a minimal prime of I and $\mathrm{d}im(P_{i+1}) = \mathrm{d}im(P_i) + 1$ for all $i = 0, \ldots, k-1$.

Exercise 7. Compute the Scarf complexex of all the generic deformations of I^3 where $I = \langle x,\ y,\ z\rangle$. Also develop a *CoCoA* function *MakeGeneric* that will deform a given monomial ideal to a generic monomial ideal. Can you modify your code in Exercise 5 to compute the irredundant irreducible decomposition of a (not necessarily generic) monomial ideal I?

Exercise 8. In Lecture 4 we saw that a generic monomial ideal I in $k[x, y, z, w]$ where every pair of generators form an edge of Δ_I can have at most 12 minimal generators. Can you give two such extremal monomial ideals with non-isomorphic Scarf complexes? A more difficult question: can you classify all Scarf complexes of these extremal monomial ideals?