Properties of Powers of Ideals

Ayesha Asloob Qureshi

(Based on joint work with Jürgen Herzog)

1 Abstract

We introduce the concept of strong persistence and show that it implies persistence regarding the associated prime ideals of the powers of ideals. We also show that strong persistence is equivalent to a condition on powers of ideals studied by Ratliff. Furthermore, we give an upper bound for the depth of powers of monomial ideals in terms of their linear relation graph and apply this to show that the index of depth stability and the index of stability for the associated prime ideals of polymatroidal ideals is bounded by their analytic spread.

2 Algebraic background and history

Let \( I \) be an ideal in a Noetherian ring \( R \). The story begins with a question of Ratliff who in the 70’s asked:

**What happens to \( \lambda_s(I^n) \) as \( n \) gets large?**

It is a general phenomenon that algebraic and homological properties of \( I \) stabilize for large \( n \).

Brodmann in 1979 showed that \( \lambda_s(I^n) \) stabilizes for large \( n \). The smallest integer for which \( \lambda(I^n) \) stabilizes is called the index of stability of \( I \) and denoted by \( \mathrm{dstab}(I) \).

3 Definitions

(1) An ideal \( I \) is said to satisfy the persistence property if \( \lambda_s(I^n) \subseteq \lambda_s(I^{n+1}) \subseteq \cdots \).

(2) Let \( P \) be a prime ideal containing \( I \). We say that \( I \) satisfies the strong persistence property with respect to \( P \) if for all \( I \) and all \( j \in \{1, \ldots, m_i\} \setminus \{j\} \), there exists \( g \in I_j \) such that \( [g] \notin \mathcal{I}^j \). The ideal \( I \) is said to satisfy the strong persistence property if it satisfies the strong persistence property for all \( P \) containing \( I \).

(3) An ideal \( I \) is said to satisfy Ratliff condition if \( I^{n+1} - I^n \) for all \( k \). Ratliff showed in [8] that this condition is satisfied for any normal ideal, and that \( I^{n+1} - I^n \) for all \( k \) in general.

**Theorem 3.1.** The ideal \( I \subseteq R \) satisfies the strong persistence property if and only if \( I^{n+1} - I^n \) for all \( k \).

4 Polymatroidal Ideals

The set of bases of a polymatroid of rank \( d \) based on \( [n] \) is a set \( B \subseteq 2^n \) of integer vectors \( a = (a(1), \ldots, a(n)) \) with non-negative entries satisfying the following conditions:

(i) \( |a| = \sum a(i) = d \) for all \( a \in B \).

(ii) (Exchange property) For all \( a, b \in B \) for which \( a(i) > b(i) \) for some \( i \), there exists \( j \in [n] \) such that \( b(j) > a(j) \) and \( a + i - j \in B \). Here \( e_i \) denotes the canonical \( i \)-th unit vector.

**Lemma 5.5.** Let \( I \) be a monomial ideal and \( \Gamma \) be the linear relation graph of \( I \). Suppose \( \Gamma \) has \( r \) vertices and \( s \) connected components. Then

\[ \ell(I) \geq r - s + 1, \]

and equality holds if \( I \) is a polymatroidal ideal.

**Conjecture 5.6.** In general the indices \( \astab(I) \) and \( \lambda(I) \) are unrelated, as shown by examples given in [4]. On the other hand the evidence of all known examples we conjecture that \( \astab(I) = \lambda(I) \) for all polymatroidal ideals.

References


International Center of Theoretical Physics
aqureshi@ictp.it