Overview

General Involutive Bases
- Shortcomings of Gröbner Bases
- Involutive Divisions with Examples
- Involutive Bases: The Monomial Case
- Involutive Bases: The Polynomial Case
- Elementary Properties

Basic Algorithms
- Pommaret Bases and $\delta$-Regularity
- Combinatorial Decompositions and Applications
- Syzygy Theory and Applications
Basic References

- V.P. Gerdt, Yu.A. Blinkov: *Involutive Bases of Polynomial Ideals*
- V.P. Gerdt, Yu.A. Blinkov: *Minimal Involutive Bases*
- W.M. Seiler: *A combinatorial approach to involution and δ-regularity I & II*
  AAECC 20 (2009) 207–338
- W.M. Seiler: *Involution — The Formal Theory of Differential Equations and its Applications in Computer Algebra*
  Springer-Verlag 2009 (Chapts. 2–5)
Gröbner Bases vs. Involutive Bases

Shortcomings of Gröbner bases:

- definition not intrinsic:
  - arbitrary choice of variables and term order
  - in principle: \( \mathcal{P} \cong \mathcal{S}\mathcal{V} \) for \( n \)-dimensional \( \mathbb{k} \)-linear space \( \mathcal{V} \) with basis \( \{x_1, \ldots, x_n\} \)
  - change of basis or term order can affect Gröbner basis drastically

- purely technical definition \( \leadsto \) algebraic properties of ideal hardly enter

- determination of algebraic structure of ideal requires additional computations (often several Gröbner bases)
Example: consider

\[ \mathcal{I} = \langle z^8 - wxy^6, y^7 - x^6z, yz^7 - wx^7 \rangle \triangleleft \mathbb{K}[w, x, y, z] \]

(reduced) Gröbner basis for degrevlex

Consider \( \mathcal{I} \) as ideal in \( \mathbb{K}[w, y, x, z] \):

(reduced) Gröbner basis for degrevlex

\[ \{ y^7 - x^6z, yz^7 - wx^7, z^8 - wxy^6, y^8z^6 - wx^{13}, \\
y^{15}z^5 - wx^{19}, y^{22}z^4 - wx^{25}, y^{29}z^3 - wx^{31}, \\
y^{36}z^2 - wx^{37}, y^{43}z - wx^{43}, y^{50} - wx^{49} \} \]
Involution Bases:

- Special type of (generally *non-reduced*) Gröbner bases with additional *combinatorial* properties.
- Provide alternative *algorithm* for construction of Gröbner bases (often *superior* to Buchberger algorithm).
- *Pommaret bases* for *degrevlex* particularly interesting for applications in *algebraic geometry* → many characteristics (e.g. degree of basis) *intrinsically* determined by *homological* invariants of ideal.
- *Pommaret bases* not only *computationally* of interest; allow for *theoretical* applications, too.
History

Differential Equations

Jacobi ≤1862
Riquier 1890–1910
(Cartan 1900–1930)
Janet 1920

Commutative Algebra

(Gröbner *1899)
(Gordan 1900)
**Differential Equations**

- Jacobi 1862
- Riquier 1890–1910
- (Cartan 1900–1930)
- Janet 1920
- Spencer 1962
- Quillen 1964
- Goldschmidt 1965
- Pommaret 1978

**Commutative Algebra**

- (Gröbner *1899)
- (Gordan 1900)
- Rees 1956 ($P$)
- Hironaka 1964/77 ($P$)
- Buchberger 1965
- Grauert 1972 ($P$)
- Stanley 1978
- Baclawski/Garcia 1981
History

Differential Equations

Jacobi ≤1862
Riquier 1890–1910
(Cartan 1900–1930)
Janet 1920
Spencer 1962
Quillen 1964
Goldschmidt 1965
Pommaret 1978

Commutative Algebra

(Gröbner *1899)
(Gordan 1900)

Rees 1956 (P)
Hironaka 1964/77 (P)
Buchberger 1965
Grauert 1972 (P)
Stanley 1978
Baclawski/Garcia 1981
Amasaki 1990 (P)
Wu 1991
Malgrange, Seiler (2002)
Notations and Conventions

- field $\mathbb{k}$ of arbitrary characteristic
  (later: “sufficiently large” for Pommaret bases)
- set of variables $X = \{x_1, \ldots, x_n\}$
  (ordering of variables relevant for many purposes!)
- polynomial ring $\mathcal{P} = \mathbb{k}[X] \rightsquigarrow$ ideals $\mathcal{I} \triangleleft \mathcal{P}$, submodules $\mathcal{M} \subseteq \mathcal{P}^m$ of free $\mathcal{P}$-module
- subset of variables $X' \subseteq X \rightsquigarrow \mathbb{T}(X')$ monoid of terms containing only variables in $X'$
- multi indices $\mu, \nu \in \mathbb{N}_0^n \rightsquigarrow$ terms $x^\mu, x^\nu \in \mathbb{T}(X)$
**Caution:** the following conventions are *non-standard*; the standard ones are obtained by reverting the ordering of the variables

\[ x_1, x_2, \ldots, x_n \quad \leftrightarrow \quad x_n, \ldots, x_2, x_1 \]

- \( \mu = [\mu_1, \ldots, \mu_n] \leadsto \text{class } \text{cls } \mu = \min \{ k \mid \mu_k \neq 0 \} \)
- **lexicographic term order**
  \( \mu \prec_{\text{lex}} \nu \iff \text{last non-vanishing entry of } \mu - \nu \text{ negative} \)
- **degree reverse lexicographic term order** (for \( |\mu| = |\nu| \))
  \( \mu \prec_{\text{degrevlex}} \nu \iff \text{first non-vanishing entry of } \mu - \nu \text{ positive} \)
- **degrevlex** only **class respecting** term order

**Lemma:** Assume \( \prec \) degree compatible term order and for all **homogeneous** polynomials \( f \) and all \( 1 \leq k \leq n \)

\[ \text{lt } f \in \langle x_1, \ldots, x_k \rangle \iff f \in \langle x_1, \ldots, x_k \rangle . \]

Then \( \prec = \prec_{\text{degrevlex}} \).
Involutive Divisions

**Basic idea:** every generator in involutive basis may only be multiplied by polynomials in its *multiplicative variables* $\leadsto$ *two ingredients* required:

- *term order* (like any Gröbner basis)
- *involutive division* for assignment of multiplicative variables (based on the leading terms of the generators)

**Difficulty:** assignment is generally made “context sensitive”: multiplicative variables for term $t \in \mathbb{T}(X)$ not absolutely defined, but $t$ must always be considered in “context” of finite set $\mathcal{T} \subseteq \mathbb{T}(X)$ (say, all leading terms in a basis) containing $t$
Involution Definitions

**Def:** involutive division $L$ on $\mathbb{T}(X)$ $\leadsto$

rule to assign to every term $t$ in any finite set $\mathcal{T} \subset \mathbb{T}(X)$ multiplicative variables $X_L, T(t)$ and thus involutive cone $C_{L, T}(t) = \mathbb{T}(X_{L, T}(t)) \cdot t$

such that:

(i) $s, t \in \mathcal{T}$ with $C_{L, T}(s) \cap C_{L, T}(t) \neq \emptyset \implies C_{L, T}(s) \subseteq C_{L, T}(t)$ or vice versa

(ii) $S \subseteq \mathcal{T} \implies \forall s \in S : X_{L, T}(s) \subseteq X_{L, S}(s)$

**Def:** $s \in \mathbb{T}(X)$ involutively divisible by $t \in \mathcal{T}$ (written $t \mid_{L, T} s$) $\leadsto$

$s \in C_{L, T}(t)$
Example: *Thomas division* \( T \)

\[ x_i \in X_{T,T}(t) \iff \deg_{x_i} t = \max \{ \deg_{x_i} s : s \in T \} \]

(independent of ordering of variables; not relevant for practice; sometimes theoretically useful)
**Example:** *Thomas division* $T \rightsquigarrow$

\[
x_i \in X_{T,T}(t) \iff \deg_{x_i} t = \max \{\deg_{x_i} s \mid s \in T\}
\]

(independent of ordering of variables; not relevant for practice; sometimes theoretically useful)

**Example:** *Janet division* $J \rightsquigarrow$

given $T \subset \mathbb{T}(X)$, introduce for arbitrary $d_i \in \mathbb{N}_0^n$ subsets

\[
(d_k, \ldots, d_n) = \{x^\mu \in T \mid \mu_k = d_k, \ldots, \mu_n = d_n\} \subseteq T
\]

then for term $t = x^\mu \in T$

- $x_n \in X_{J,T}(t) \iff \deg_{x_n} t = \max \{\deg_{x_n} s \mid s \in T = ()\}$
- for $1 \leq k < n$: $x_k \in X_{J,T}(t) \iff \deg_{x_k} t = \max \{\deg_{x_k} s \mid s \in (\mu_{k+1}, \ldots, \mu_n)\}$

(“refinement” of Thomas division; depends on ordering of variables)
Example: Pommaret division $P \rightsquigarrow$

$t = x^\mu \in \mathcal{T}, \text{cls } \mu = k \implies X_P(t) = \{x_1, \ldots, x_k\}$

(depending on ordering of variables, too)

global division: no dependence on $\mathcal{T}$
**Example:** *Pommaret division* \( P \)  
\[ t = x^\mu \in \mathcal{T}, \text{cls} \mu = k \implies X_P(t) = \{x_1, \ldots, x_k\} \]  
(depends on ordering of variables, too)

**Global division:** no dependence on \( \mathcal{T} \)

**Example:** bizarre global division on \( \mathbb{T}(x, y, z) \)

\[ X_L(1) = \{x, y, z\} \]
\[ X_L(x) = \{x, z\}, \quad X_L(y) = \{x, y\}, \quad X_L(z) = \{y, z\}, \]
\[ X_L(t) = \emptyset \text{ for all other } t \in \mathbb{T}(x, y, z) \]
Example: $\mathcal{T} = \{ z^3, y^2 z, yz^2, xz^2, xyz \} \subset \mathbb{C}(x, y, z)$

multiplicative variables for different involutive divisions

<table>
<thead>
<tr>
<th></th>
<th>$z^3$</th>
<th>$y^2 z$</th>
<th>$yz^2$</th>
<th>$xz^2$</th>
<th>$xyz$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>$z$</td>
<td>$y$</td>
<td>$-$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>J</td>
<td>$x, y, z$</td>
<td>$x, y$</td>
<td>$x, y$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>P</td>
<td>$x, y, z$</td>
<td>$x, y$</td>
<td>$x, y$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
</tbody>
</table>

(note: despite very different definitions, Janet and Pommaret division yield here same multiplicative variables $\implies$ more in Lecture 3!)

W.M. Seiler: Involution Bases I – 7
Monomial Involutive Bases

**Def**: $\mathcal{T} \subset \mathbb{T}(X)$ finite, involutive division $L$

- involutive span of $\mathcal{T}$ $\leadsto \langle \mathcal{T} \rangle_L = \langle \bigcup_{t \in \mathcal{T}} C_L,\mathcal{T}(t) \rangle_k$
- $\mathcal{T}$ weakly involutive $\leadsto \langle \mathcal{T} \rangle_L = \langle \mathcal{T} \rangle$
- $\mathcal{T}$ (strongly) involutive $\leadsto$ additionally
  \[
  \forall t \neq t' \in \mathcal{T} : C_L,\mathcal{T}(t) \cap C_L,\mathcal{T}(t') = \emptyset
  \]
  (provides $k$-linear direct sum decomposition of ideal $\langle \mathcal{T} \rangle$!)

- $\mathcal{T} \subseteq \hat{\mathcal{T}}$ (finite) weakly involutive completion $\leadsto \langle \hat{\mathcal{T}} \rangle_L = \langle \mathcal{T} \rangle$

- $\mathcal{T}$ (weakly) involutive basis of monomial ideal $\mathcal{I} \triangleleft \mathcal{P}$ $\leadsto \mathcal{T}$ (weakly) involutive and $\langle \mathcal{T} \rangle_L = \mathcal{I}$
Def: \( \mathcal{T} \subset \mathbb{T}(X) \) finite, involutive division \( L \)

- involutive span of \( \mathcal{T} \) \( \rightsquigarrow \) \( \langle \mathcal{T} \rangle_L = \langle \bigcup_{t \in \mathcal{T}} C_L,\mathcal{T}(t) \rangle_k \)

- \( \mathcal{T} \) weakly involutive \( \rightsquigarrow \) \( \langle \mathcal{T} \rangle_L = \langle \mathcal{T} \rangle \)

- \( \mathcal{T} \) (strongly) involutive \( \rightsquigarrow \) additionally

\[
\forall t \neq t' \in \mathcal{T} : C_L,\mathcal{T}(t) \cap C_L,\mathcal{T}(t') = \emptyset
\]

(provides \( k \)-linear direct sum decomposition of ideal \( \langle \mathcal{T} \rangle \) !)

- \( \mathcal{T} \subseteq \hat{\mathcal{T}} \) (finite) weakly involutive completion \( \rightsquigarrow \) \( \langle \hat{\mathcal{T}} \rangle_L = \langle \mathcal{T} \rangle \)

- \( \mathcal{T} \) (weakly) involutive basis of monomial ideal \( \mathcal{I} \triangleleft \mathcal{P} \) \( \rightsquigarrow \) \( \mathcal{T} \) (weakly) involutive and \( \langle \mathcal{T} \rangle_L = \mathcal{I} \)
Monomial Involutive Bases

**Def:** \( \mathcal{T} \subset \mathbb{T}(X) \) finite, involutive division \( L \)

- involutive span of \( \mathcal{T} \) \( \leadsto \) \( \langle \mathcal{T} \rangle_L = \langle \bigcup_{t \in \mathcal{T}} C_{L,T}(t) \rangle_k \)
- \( \mathcal{T} \) weakly involutive \( \leadsto \) \( \langle \mathcal{T} \rangle_L = \langle \mathcal{T} \rangle \)
- \( \mathcal{T} \) (strongly) involutive \( \leadsto \) additionally
  \[ \forall t \neq t' \in \mathcal{T} : C_{L,T}(t) \cap C_{L,T}(t') = \emptyset \]
  (provides \( k \)-linear direct sum decomposition of ideal \( \langle \mathcal{T} \rangle \)!) 

- \( \mathcal{T} \subseteq \hat{\mathcal{T}} \) (finite) weakly involutive completion \( \leadsto \) \( \langle \hat{\mathcal{T}} \rangle_L = \langle \mathcal{T} \rangle \)
- \( \mathcal{T} \) (weakly) involutive basis of monomial ideal \( \mathcal{I} \triangleleft \mathcal{P} \) \( \leadsto \)
  \( \mathcal{I} \) (weakly) involutive and \( \langle \mathcal{T} \rangle_L = \mathcal{I} \)
Monomial Involutive Bases

Def: \( \mathcal{T} \subset \mathbb{T}(X) \) finite, involutive division \( L \)

- involutive span of \( \mathcal{T} \) \( \leadsto \) \( \langle \mathcal{T} \rangle_L = \bigcup_{t \in \mathcal{T}} \mathcal{C}_{L,T}(t) \) \( \subseteq \mathbb{k} \)

- \( \mathcal{T} \) weakly involutive \( \leadsto \) \( \langle \mathcal{T} \rangle_L = \langle \mathcal{T} \rangle \)

- \( \mathcal{T} \) (strongly) involutive \( \leadsto \) additionally

\[
\forall t \neq t' \in \mathcal{T} : \mathcal{C}_{L,T}(t) \cap \mathcal{C}_{L,T}(t') = \emptyset
\]

(provides \( \mathbb{k} \)-linear direct sum decomposition of ideal \( \langle \mathcal{T} \rangle \)!) \( \subseteq \mathbb{k} \)

- \( \mathcal{T} \subseteq \hat{\mathcal{T}} \) (finite) weakly involutive completion \( \leadsto \) \( \langle \hat{\mathcal{T}} \rangle_L = \langle \mathcal{T} \rangle \)

- \( \mathcal{T} \) (weakly) involutive basis of monomial ideal \( \mathcal{I} \subset \mathcal{P} \) \( \leadsto \)

\( \mathcal{T} \) (weakly) involutive and \( \langle \mathcal{T} \rangle_L = \mathcal{I} \)
Def: $\mathcal{T} \subset \mathbb{T}(X)$ finite, involutive division $L$

- involutive span of $\mathcal{T} \rightsquigarrow \langle \mathcal{T} \rangle_L = \left\langle \bigcup_{t \in \mathcal{T}} C_{L,\mathcal{T}}(t) \right\rangle_k$

- $\mathcal{T}$ weakly involutive $\rightsquigarrow \langle \mathcal{T} \rangle_L = \langle \mathcal{T} \rangle$

- $\mathcal{T}$ (strongly) involutive $\rightsquigarrow$ additionally
  \[
  \forall t \neq t' \in \mathcal{T} : C_{L,\mathcal{T}}(t) \cap C_{L,\mathcal{T}}(t') = \emptyset
  \]
  (provides $\mathbb{k}$-linear direct sum decomposition of ideal $\langle \mathcal{T} \rangle$!)

- $\mathcal{T} \subseteq \hat{\mathcal{T}}$ (finite) weakly involutive completion $\rightsquigarrow \langle \hat{\mathcal{T}} \rangle_L = \langle \mathcal{T} \rangle$

- $\mathcal{T}$ (weakly) involutive basis of monomial ideal $\mathcal{I} \triangleleft \mathcal{P} \rightsquigarrow \mathcal{T}$ (weakly) involutive and $\langle \mathcal{T} \rangle_L = \mathcal{I}$

W.M. Seiler: Involutive Bases I – 8
Monomial Involutive Bases

Overview
Basic References
Gröbner Bases vs. Involutive Bases
History
Notations and Conventions
Involutive Divisions
Monomial Involutive Bases
Polynomial Involutive Bases
Monomial Involutive Bases

Overview
Basic References
Gröbner Bases vs. Involutive Bases
History
Notations and Conventions
Involutive Divisions
Monomial Involutive Bases
Polynomial Involutive Bases
\[ \mathfrak{k}-\text{linear isomorphism: } \quad (Stanley \text{ decomposition} \quad \rightsquigarrow \quad \text{Lecture 4}) \]

\[ \mathcal{I} = \langle x^2, y^2 \rangle \cong \mathfrak{k}[x, y] \cdot y^2 \oplus \mathfrak{k}[x] \cdot x^2 \oplus \mathfrak{k}[x] \cdot x^2 y \]
Prop: \( \mathcal{T} \) weakly involutive basis \( \implies \) \( \exists \mathcal{T}' \subseteq \mathcal{T} \) strongly involutive basis

Proof:

\( \mathcal{T} \) weakly but not strongly involutive basis \( \implies \)

\[ \exists s \neq t \in \mathcal{T} : C_{L,T}(s) \cap C_{L,T}(t) \neq \emptyset \] \( (i) \)

(wlog) \[ C_{L,T}(s) \subseteq C_{L,T}(t) \] \( \leadsto \) set \( \mathcal{T}' = \mathcal{T} \setminus \{s\} \)

\( (ii) \) \( \mathcal{T}' \) still weakly involutive basis \( \leadsto \) iterate
Example: minimal basis of irreducible monomial ideal of the form
\[ \mathcal{T} = \{ x_{i_1}^{\ell_{i_1}}, \ldots, x_{i_2}^{\ell_{i_2}}, x_{i_r}^{\ell_{i_r}} \} \]
with \( 1 \leq r \leq n \) generators sorted according to \( i_r > \cdots > i_2 > i_1 \)

\( \mathcal{T} \) has finite Pommaret completion \( \hat{\mathcal{T}} \iff i_r = n, \quad i_{r-1} = n - 1, \ldots, \quad i_1 = n - r + 1 \) (i.e. no “gaps”)  
(note: always achievable by renumbering!)

completion \( \hat{\mathcal{T}} \) consists then of all terms of the form  
\[ x_{i_{j_1}}^{\ell_{i_{j_1}}}, x_{i_{j_1}+1}^{k_{j_1}+1}, \ldots, x_{i_{p_{j}}+1}^{k_{p_{j}}+1} \]  
with \( \forall m > i_{j_1} : k_m < \ell_m \)

(thus maximal degree of generator:
\[ 1 - r + \sum_{j=1}^{r} \ell_{i_j} \])

if “gap” exists at position \( m \leadsto \) no bound for \( k_m \)
Monomial Involutive Bases

**Def:** involutive division \( L \) Noetherian \( \iff \)
every finite \( \mathcal{T} \subset \mathbb{T}(X) \) possesses involutive completion

**Lemma:** Janet division Noetherian

**Proof:** \( s = \text{lcm} \mathcal{T} \implies \hat{\mathcal{T}} = \{ t \in \langle \mathcal{T} \rangle : t \mid s \} \) Janet basis of \( \langle \mathcal{T} \rangle \)

**Remark:** Pommaret division *not* Noetherian by example above
simplest counterexample: \( \mathcal{T} = \{ xy \} \subset \mathbb{T}(x, y) \)
ideal \( \langle \mathcal{T} \rangle \) contains *only* terms of class 1 \( \implies \)
infinite Pommaret “basis” \( \{ xy^k \mid k \in \mathbb{N} \} \)
Def: \( \mathcal{I} \triangleleft \mathcal{P} \) polynomial ideal, finite set \( \mathcal{H} \subset \mathcal{I} \) (term order \( \prec \), involutive division \( L \))

- \( \mathcal{H} \) weakly involutive basis of \( \mathcal{I} \)  \( \leadsto \)
  \( \text{lt} \mathcal{H} \) weakly involutive basis of \( \text{lt} \mathcal{I} \)

- \( \mathcal{H} \) involutive basis of \( \mathcal{I} \)  \( \leadsto \)
  \( \text{lt} \mathcal{H} \) involutive basis of \( \text{lt} \mathcal{I} \) and all leading terms pairwise distinct

Lemma: \( \mathcal{H} \) (weakly) involutive basis  \( \Rightarrow \) \( \mathcal{H} \) Gröbner basis
Def: \( I \triangleleft \mathcal{P} \) polynomial ideal, finite set \( H \subset I \) (term order \( \prec \), involutive division \( L \))

- \( H \) *weakly involutive basis* of \( I \) \( \leadsto \)
  - \( \text{lt} \) \( H \) weakly involutive basis of \( \text{lt} \) \( I \)

- \( H \) *involutive basis* of \( I \) \( \leadsto \)
  - \( \text{lt} \) \( H \) involutive basis of \( \text{lt} \) \( I \) and all leading terms pairwise distinct

Lemma: \( H \) (weakly) involutive basis \( \implies \) \( H \) Gröbner basis

Prop: \( H \) weakly involutive basis \( \implies \) \( \exists H' \subset H \) strongly involutive basis

Proof: as in monomial case

(weakly involutive bases required for generalisations like semigroup orders or polynomials over rings where strongly involutive bases generally do not exist)
Def: \( F \subset P \) finite set of polynomials

- **multiplicative variables** \( \leadsto \forall f \in F : X_{L,F,\prec}(f) = X_{L,\text{lt } F}(\text{lt } f) \)
- **involution span** \( \leadsto \langle F \rangle_L = \sum_{f \in F} k[X_{L,F,\prec}(f)] \cdot f \)
Polynomial Involutive Bases

Def: \( \mathcal{F} \subset \mathcal{P} \) finite set of polynomials

- multiplicative variables \( \rightsquigarrow \forall f \in \mathcal{F} : X_L,\mathcal{F},\prec(f) = X_L,\text{lt}\mathcal{F}(\text{lt} f) \)
- involutive span \( \rightsquigarrow \langle \mathcal{F} \rangle_L = \sum_{f \in \mathcal{F}} \mathbb{K}[X_L,\mathcal{F},\prec(f)] \cdot f \)

Def: \( \mathcal{F} \subset \mathcal{P} \) finite set of polynomials, \( g \in \mathcal{P} \) further polynomial

- \( g \) involutely head reducible wrt \( \mathcal{F} \) \( \rightsquigarrow \exists f \in \mathcal{F} : \text{lt} f \big|_{L,\text{lt}\mathcal{F}} \text{lt} g \)
- \( g \) in involutive normal form wrt \( \mathcal{F} \) \( \rightsquigarrow \text{supp} g \cap \langle \text{lt} \mathcal{F} \rangle_L = \emptyset \)
- \( \mathcal{F} \) involutely head autoreduced \( \rightsquigarrow \nexists f_1 \neq f_2 \in \mathcal{F} : \text{lt} f_1 \big|_{L,\text{lt}\mathcal{F}} \text{lt} f_2 \)

(algorithms for computing involutive normal forms or for involutive head autoreduction are obtained by trivial modifications of usual algorithms)
Theorem: The following are equivalent:

(i) \( \mathcal{H} \) weakly involutive basis of ideal \( \mathcal{I} \preceq \mathcal{P} \)

(ii) every \( f \in \mathcal{I} \) possesses involutive standard representation

\[ f = \sum_{h \in \mathcal{H}} P_h \cdot h \quad \text{with} \quad \text{lt} (P_h h) \preceq \text{lt} f \land P_h \in \mathbb{k}[X_L, \mathcal{H}, <(h)] \]
Theorem: The following are equivalent:

(i) \( \mathcal{H} \) weakly involutive basis of ideal \( \mathcal{I} \triangleleft \mathcal{P} \)

(ii) every \( f \in \mathcal{I} \) possesses involutive standard representation

\[
f = \sum_{h \in \mathcal{H}} P_h \cdot h \quad \text{with} \quad \text{lt} \ (P_h h) \preceq \text{lt} \ f \land P_h \in \mathbb{K}[X_L, \mathcal{H}, \prec(h)]
\]

\( \mathcal{H} \) strongly involutive basis \iff\ involutive standard representation unique
Theorem: The following are equivalent:

(i) $\mathcal{H}$ weakly involutive basis of ideal $\mathcal{I} \triangleleft \mathcal{P}$

(ii) every $f \in \mathcal{I}$ possesses involutive standard representation

$$f = \sum_{h \in \mathcal{H}} P_h \cdot h \quad \text{with} \quad \text{lt}(P_h h) \preceq \text{lt}f \land P_h \in \mathbb{K}[X_L, H, \prec(h)]$$

$\mathcal{H}$ strongly involutive basis $\iff$ involutive standard representation unique

Proof:

- "⇒" compute involutive normal form
- "⇐" leading terms show that $\langle \text{lt} \mathcal{H} \rangle_L = \text{lt} \mathcal{I}$

- uniqueness follows from direct sum decomposition
  (at each step of normal form computation only one possible divisor!)
Polynomial Involutively Bases

**Theorem:** The following are equivalent:

(i) \( \mathcal{H} \) weakly involutive basis of ideal \( \mathcal{I} \triangleleft \mathcal{P} \)

(ii) every \( f \in \mathcal{I} \) possesses **involutively standard representation**

\[
f = \sum_{h \in \mathcal{H}} P_h \cdot h \quad \text{with} \quad \text{lt} (P_h h) \preceq \text{lt} f \land P_h \in \mathbb{k}[X_L, \mathcal{H}, \prec(h)]
\]

\( \mathcal{H} \) strongly involutive basis \( \iff \) involutive standard representation **unique**

**Corollary:**

- \( \mathcal{H} \) weakly involutive basis \( \implies \langle \mathcal{H} \rangle_{L, \prec} = \mathcal{I} \)
- \( \mathcal{H} \) strongly involutive basis \( \implies \mathcal{I} = \bigoplus_{h \in \mathcal{H}} \mathbb{k}[X_L, \mathcal{H}, \prec(h)] \cdot h \)

(\( \mathbb{k} \)-linear direct sum decomposition)
**Theorem:** The following are equivalent:

(i) $\mathcal{H}$ weakly involutive basis of ideal $\mathcal{I} \triangleleft \mathcal{P}$

(ii) every $f \in \mathcal{I}$ possesses *involutive standard representation*

\[ f = \sum_{h \in \mathcal{H}} P_h \cdot h \quad \text{with} \quad \text{lt}(P_h h) \preceq \text{lt} f \wedge P_h \in \mathbb{K}[X_L, \mathcal{H}, \prec(h)] \]

$\mathcal{H}$ *strongly* involutive basis $\iff$ involutive standard representation *unique*

**Caution:** $\langle \mathcal{H} \rangle_{L, \prec} = \mathcal{I} \iff \mathcal{H}$ weakly involutive basis

**Example:** $\mathcal{H} = \{y^2, y^2 + x^2\}$ and $L = J$ (Janet division)

$\langle \mathcal{H} \rangle_J, \prec = \mathcal{I} = \langle \mathcal{H} \rangle$ but $x^2 \in \text{lt} \mathcal{I} \setminus \langle \text{lt} \mathcal{H} \rangle_J$
Polynomial Involution Bases

Example: \( \prec \) degree compatible order

\[ \mathcal{F} = \{ f_1 = z^2 - xy, \ f_2 = yz - x, \ f_3 = y^2 - z \} \subset \mathbb{k}[x, y, z] \]

\( \text{\textbullet} \) \( \deg f_1 = z^2 \implies \mathcal{F} \) Janet basis

\( \text{\textbullet} \) \( \deg f_1 = xy \implies f_4 = zf_1 + xf_2 = z^3 - x^2 \)
   has no standard representation

\( \square \) \( \mathcal{F} \cup \{ f_4 \} \) Gröbner basis, but not Janet basis

\( \square \) \( \mathcal{F} \cup \{ f_4, \ f_5 = zf_2 \} \) Janet basis

Involution bases are generally \textit{non-reduced} Gröbner bases!
Prop: \( \mathcal{F} \subset \mathcal{P} \) finite, involutively head autoreduced set \( \Longrightarrow \) involutive normal form of any polynomial \( g \in \mathcal{P} \) wrt \( \mathcal{F} \) unique

Proof: \( \mathcal{F} \) induces direct sum decomposition of \( \langle \mathcal{F} \rangle_{L,<} \) \( \sim \sim \)

claim clear for \( g \in \langle \mathcal{F} \rangle_{L,<} \): always involutive normal form \( 0 \) \( \sim \sim \)

\( g_1, g_2 \) two different involutive normal forms of \( g \) \( \Longrightarrow \)

\( g_1 - g_2 \in \langle \mathcal{F} \rangle_{L,<} \) in involutive normal form
**Prop:** $\mathcal{F} \subset \mathcal{P}$ finite, involutively head autoreduced set $\implies$ involutive normal form of any polynomial $g \in \mathcal{P}$ wrt $\mathcal{F}$ unique

**Proof:** $\mathcal{F}$ induces direct sum decomposition of $\langle \mathcal{F} \rangle_{L,\prec}$, claim clear for $g \in \langle \mathcal{F} \rangle_{L,\prec}$: always involutive normal form $0 \implies g_1, g_2$ two different involutive normal forms of $g \implies g_1 - g_2 \in \langle \mathcal{F} \rangle_{L,\prec}$ in involutive normal form

**Prop:** $\mathcal{F} \subset \mathcal{P}$ finite, weakly involutive set $\implies$ involutive and usual normal form of any polynomial $g \in \mathcal{P}$ wrt $\mathcal{F}$ coincide

**Proof:**
- involutive normal form wrt weakly involutive basis unique (similar argument as above)
- usual normal form wrt Gröbner basis unique
- weakly involutive basis is Gröbner basis
- usual normal form trivially involutive normal form