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Castelnuovo-Mumford regularity and applications

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Bounds in terms of the degrees of generators

We have introduced two measures of the complexity of an homogeneous ideal $I \subseteq P = k[x_1, \dots, x_n]$:

- $d(I)$ the maximum degree of a polynomial in a minimal system of generators of I (actually of the generators of $gin_{revlex}(I)$)
- $reg(I)$: the maximum degree of the syzygies in a minimal free resolution of I

Question How much bigger can $reg(I)$ be than $d(I)$?

Obviously:

$$d(I) \leq reg(I)$$

Conjecture (Bayer '82):

$$reg(I) \leq d(I)^{2^{n-1}}$$

Bounds in terms of the degrees of generators

Giusti-Galligo (84) : If $\operatorname{char} k = 0$ then

$$reg(I) \leq (2d(I))^{2^{n-2}}$$

There are examples with very large regularity (Mayr-Mayer), see tutorial.

The regularity can really be doubly exponential in the degrees of the generators and the number of the variables.

Koh (98) : For each integer $r \geq 1$ there exists an ideal $I_r \subseteq P = k[x_1, \ldots, x_n]$ with $n = 22r$ generated by quadrics such that

$$reg(I_r) \geq 2^{2^{r-1}}$$

These examples are highly non reduced (see also Giamo (2004) for a way of making reduced examples).

Bounds in terms of the degrees of generators

Bayer-Mumford in any characteristic

$$reg(I) \leq (2d(I))^{(n-1)!}$$

In the same paper they asked whether Giusti-Galligo's bound holds in any characteristic

Caviglia-Sbarra: If $\mathrm{ht}(I) = c < n$ and I is generated in degree $\leq d$, then

$$reg(I) \leq (d^c + (d-1)c + 1)^{2^{n-c-1}}$$

As a consequence we may deduce

- $n = 2$ $reg(I) \leq 2d$
- $n \geq 3$ $reg(I) \leq (d^2 + 2d - 1)^{2^{n-3}} \leq (2d)^{2^{n-2}}$
  (the worst case is $\mathrm{ht}(I) = 2$.)

Bounds in terms of the degrees of generators

Problem.[Peeva-Stillman] Let $d_1 \geq d_2 \geq \ldots$ the degrees of the elements in a minimal system of generators of I. Set $c = ht(I)$, find conditions on I such that

$$\mathrm{reg}(I) \leq d_1 + \cdots + d_c - c + 1$$

Exercise.
Let $I \subseteq P = k[x_1, \ldots, x_n]$, $\dim P/I = 0$, I is generated in degree $\leq d$, then

$$reg(I) \leq n(d-1) + 1$$

Sjögren : The previous fact holds assuming $\dim P/I \leq 1$.

Bounds in terms of the degrees of generators

For smooth (or nearly smooth) varieties there are much better bounds, linear in the degrees of the generators and in the number of the variables.

Theorem (Bertram-Ein-Lazarsfeld and Chardin-Ulrich)

Assume char $k = 0$ and $X \subseteq \mathbb{P}^r$ a smooth variety defined scheme-theoretically by equations of degree $\leq d$, then

$$reg(I(X)) \leq 1 + (d-1)r.$$

More precisely if codim $X = c$ and X is defined scheme-theoretically by equations of degree $d_1 \geq d_2 \geq \dots,$ then

$$reg(I(X)) \leq d_1 + \cdots + d_c - c + 1$$

Bounds in terms of the degrees of generators

In the line of the papers by Bertram-Ein-Lazarsfeld and Chardin-Ulrich (smooth variety defined scheme-theoretically by equations of degree $\leq d$), one can ask the following problems.

Problem.[Eisenbud] The previous result might be true for any reduced algebraic set over an algebraically closed field.

Eisenbud-Goto's Conjecture

Eisenbud-Goto Conjecture (84): If $\wp \subseteq (x_1, \ldots, x_n)^2$ is a prime homogeneous ideal, then

$$\mathrm{reg}(P/\wp) \leq e(P/\wp) - n + dimP/\wp$$

- It is proved for irreducible curves (Gruson, Lazarsfeld, Peskine '83)
- It is proved for smooth surfaces (Bayer-Mumford '93). Some more generality (Brodman'99)
- It is proved for some classes of toric varieties in codimension two (Peeva-Sturmfels '98)
- Slightly weaker bounds (still linear in the degree) for smooth varieties of dimension ≤ 6 (Kwak 2000)

There are evidence that EG conjecture should be true at least for smooth schemes in char zero (papers by Mumford, Bertram-Ein-Lazarsfeld , Chardin)

Eisenbud-Goto Conjecture

Eisenbud has conjectured that the bound of the Conjecture holds if X is reduced and connected in codimension 1.

Both the assumptions are necessary, as the following examples show:

- Two skew lines in $\mathbb{P}^3$: let

$$I = (x_0, x_1) \cap (x_2, x_3) \subseteq P = k[x_0, \ldots, x_3].$$

  In this case $e(P/I) = 2, codim = 2,$ so $reg(I) = 2 > e - codim + 1.$

- A multiple line in $\mathbb{P}^3$: let

$$I = (x_0, x_1)^2 + (x_2^d x_0 + x_3^d x_1) \subseteq P = k[x_0, \ldots, x_3].$$

  In this case $e(P/I) = 2, codim = 2,$ and
  $reg(I) = d + 1 > e - codim + 1 = 1.$

Regularity of the radical

Ravi proved that if I is a monomial ideal, then

$$reg(\sqrt{I}) \leq reg(I)$$

Problem. Find classes of ideals for which $reg(\sqrt{I}) \leq reg(I)$.

Chardin-D'Cruz produced examples where $reg(\sqrt{I})$ is the cube of $reg(I)$ (see tutorial).

Problem.(Peeva-Stillman) Is $reg(\sqrt{I})$ bounded by a (possibly polynomial) function of $reg(I)$?

Regularity of the Tangent Cone

Let $A = k[[x_1, \dots, x_n]]/I$ a local ring and let m be its maximal ideal. We define the homogeneous k-standard algebra

$$gr_m(A) = \oplus_{n \geq 0} m^n / m^{n+1}$$

which is called the associated graded ring or the tangent cone of A.

Geometric construction If A is the localization at the origin of the coordinate ring of an affine variety V passing through 0, then $gr_m(A)$ is the coordinate ring of the *tangent cone* of V, which is the cone composed of all lines that are limiting positions of secant lines to V in 0. The *Proj* of this algebra can also be seen as the *exceptional set* of the *blowing-up* of V in 0.

We have a nice presentation

$$gr_m(A) \simeq k[x_1, \dots, x_n]/I^*$$

where I^* is the ideal generated by the initial forms (w.r.t. the m-adic filtration) of the elements of I. The ideal I^* can by computed by using a slight modification of Buchberger's algorithm (see Mora, Traverso).

Example

Example

Consider the power series $A = k[[t^4, t^5, t^{11}]]$. This is a one-dimensional local domain and

$$A = k[[x, y, z]]/I \quad \text{where} \quad I = (x^4 - yz, y^3 - xz, z^2 - x^3y^2).$$

We can prove that

$$gr_m(A) = k[x, y, z]/(xz, yz, z^2, y^4)$$

We have $\dim A = \dim\ gr_m(A) = 1$, but depth $gr_m(A) = 0$.

We always have $\dim A = \dim\ gr_m(A)$, but the above example shows that

$$A \text{ Cohen-Macaulay} \not\Longrightarrow gr_m(A) \text{ Cohen-Macaulay}$$

Minimal free resolution of the tangent cone

Denote by $\mu(\ \ )$ the minimal number of generators of an ideal of A. The Hilbert function of A is, by definition

$$HF_A(n) := dim_k\ m^n/m^{n+1} = \mu(m^n)$$

for every $n \geq 0$. Hence HF_A is the Hilbert function of the homogeneous k-standard algebra

$$gr_m(A) = \oplus_{n\geq 0} m^n/m^{n+1}$$

In particular $e(A) = e(gr_m(A)),\ \dim A = \dim gr_m(A).$

Several papers have been produced concerning the following problem:

Problem: Compare the numerical invariants of the R-free minimal resolution of A ($R = k[[x_1, \dots, x_n]]$) with those of the P-free minimal graded resolution ($P = k[x_1, \dots, x_n]$) of $gr_m(A)$:

$$0 \to R^{\beta_h(I)} \to R^{\beta_{h-1}(I)} \to \cdots \to R^{\beta_0(I)} \to I \to 0$$

$$0 \to P^{\beta_s(I^*)} \to P^{\beta_{s-1}(I^*)} \to \cdots \to P^{\beta_0(I^*)} \to I^* \to 0$$

Minimal free resolution of the tangent cone

Robbiano ([R]) proved

$$\beta_i(I) \leq \beta_i(I^*)$$

In general is $<$

Example (Herzog,Rossi, Valla)

Consider $I = (x^3 - y^7, x^2y - xt^3 - z^6)$ in $R = k[[x, y, z, t]]$. Since I is a complete intersection, then a minimal free resolution of I is given by:

$$0 \to R \to R^2 \to I \to 0.$$

But we can verify that

$$I^* = (x^3, x^2y, x^2t^3, xt^6, x^2z^6, xy^9 - xz^6t^3, xy^8t^3, y^7t^9),$$

hence $\mu(I^*) = 8$ and a minimal free resolution of I^* is given by

$$0 \to P \to P^6 \to P^{12} \to P^8 \to I^* \to 0$$

Regularity of $gr_m(A)$

It is an interesting problem to study the Castelnuovo-Mumford regularity of the tangent cone of a Cohen-Macaulay local ring.

- If $gr_m(A)$ is a Cohen-Macaulay graded algebra, then

$$reg(gr_m(A)) \leq e(A) - h + 1$$

  where h is the codimension of A.

- A 1-dimensional Cohen-Macaulay then

$$reg(gr_m(A)) \leq e(A) - 1.$$

Problem. [Rossi, Trung, Valla] Let (A, m) be a local Cohen-Macaulay ring. Is $reg(gr_m(A))$ bounded by a polynomial function (possibly linear) of the multiplicity $e(A)$ and the codimension?

Srinivas-Trivedi, Rossi-Trung-Valla proved very large bounds.

References I

- Bayer D., Mumford D., What can be computed in algebraic geometry? Computational Algebraic Geometry and Commutative Algebra, D. Eisenbud and L. Robbiano (eds.), Sympos. Math. 34, Cambridge Univ. Press, Cambridge, (1993), 1-48.
- Bayer D., Stillman M., A criterion for detecting m-regularity, Invent. Math. 87 (1987), N. 1, 1-11.
- Bayer D., Stillman M. On the complexity of computing syzygies, J. Symbolic Comput. 6:2-3 (1988), 135-147.
- Bermejo I., Gimenez P., Saturation and Castelnuovo-Mumford regularity, J . Algebra 303 (2006), 592–617.
- Bertram A., Ein L., Lazarsfeld R.: Vanishing theorems, a theorem of Severi, and the equations defining projective varieties, J. Amer. Math. Soc. 4 (1991), 587-602.
- Bigatti, A., Upper bounds for the Betti numbers of a given Hilbert function. Comm. Algebra 21 (1993), no. 7, 2317–2334.

References II

- Capani A., Niesi G., Robbiano L., CoCoA, a system for doing Computations in Commutative Algebra, available via anonymus ftp from: cocoa.dima.unige.it.
- Caviglia G., Sbarra E., Characteristic-free bounds for the Castelnuovo-Mumford regularity, 2003. Available at arXiv:math.AC/0310122.
- Charalambous H., Evans G.: Problems on Betti numbers of finite length modules, in Free resolutions in commutative algebra and algebraic geometry (Sundance, UT, 1990), 25-33, Res. Notes Math. 2, Jones and Bartlett, Boston, 1992.
- Chardin M. , D'Cruz C.: Castelnuovo-Mumford regularity: examples of curves and surfaces, J. ALgebra 270 (2003), 347-360.
- Chardin, M., Some results and Questions on Castelnuovo-Mumford regularity, Syzygies and Hilbert functions, ed. by I. Peeva, Lecture Notes in Pure and Applied Mathematics, 254, Chapman Hall/CRC, Boca Raton, FL, 2007.

References III

- Conca A., Herzog J., Castelnuovo-Mumford regularity of products of ideals, Collect. Math. 54 (2003), 137-152.
- Derksen H., Sidman J.: A sharp bound for the Castelnuovo-Mumford regularity of subspace arrangements, Adv. Math. 172 (2002), 151-157.
- Eisenbud, D., Commutative algebra With a view toward algebraic geometry. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.
- Eisenbud D. , The Geometry of Syzygies, A Second Course in Commutative Algebra and Algebraic Geometry, Graduate Texts in Mathematics 229, Springer-Verlag, New York, 2005.
- Eisenbud D. (with a chapter by J. Sidman) , Lectures on the Geometry of Syzygies, Trends in Commutative Algebra MSRI Publications Volume 51, 2004.
- Eisenbud D., Goto S., Linear free resolutions and minimal multiplicity, J. Algebra 88, 1 (1984), 89-133.

References IV

- Gruson L., Lazarsfeld R., Peskine C., On a theorem of Castelnuovo and the equations defining space curves, Invent. Math. 72, 3 (1983), 491-506.
- Kreuzer M, Robbiano L., Computational Commutative Algebra 1, Springer, Heidelberg (2000).
- Kreuzer M, Robbiano L., Computational Commutative Algebra 2, Springer, Heidelberg (2005).
- Kwak S., Castelnuovo regularity for smooth subvarieties of dimensions 3 and 4", J. Algebraic Geom. 7:1 (1998), 195-206.
- Kwak S., Generic projections, the equations defining projective varieties and Castelnuovo regularity , Math. Z. 234:3 (2000), 413-434.
- Lazarsfeld R.: A sharp Castelnuovo bound for smooth surfaces, Duke Math. J. 55 (1987), 423-429.
- Mayr E., Meyer A.: The complexity of the word problems for commutative semigroups and polynomial ideals, Adv. Math. 46 (1982), 305-329.

References IV

- Peeva I., Stillman M., Open Problems on syzygies and Hilbert functions, Journal of Comm. Algebra, Vol. 1, N. 1 (2009), 159-195
- Peeva I., Sturmfels B., Syzygies of codimension 2 lattice ideals, Math. Z. 229:1 (1998), 163-194.
- Robbiano L., Coni tangenti a singolarita' razionali, Curve algebriche, Istituto di Analisi Globale, Firenze, 1981.
- Rossi M.E., Trung N.V., Valla G., Cohomological degree and Castelnuovo-Mumford regularity, Trans. Amer. Math. Soc. 355, N.5 (2003), 1773–1786.
- Rossi M.E., Trung N.V., Valla G., Castelnuovo-Mumford regularity and finiteness of Hilbert functions. Commutative algebra, 193–209, Lect. Notes Pure Appl. Math., 244, Chapman Hall/CRC, Boca Raton, FL, 2006.
- Seiler, W., A Combinatorial Approach to Involution and δ-Regularity II: Structure Analysis of Polynomial Modules with Pommaret Bases (preprint), 2009.

References V

- Srinivas V. and Trivedi V., On the Hilbert function of a Cohen-Macaulay ring, J. Algebraic Geom. 6 (1997), 733-751.
- Trivedi V., Hilbert functions, Castelnuovo-Mumford regularity and uniform Artin-Rees numbers, Manuscripta Math. 94 (1997), no. 4, 485–499.
- Trung N.V., Evaluations of initial ideals and Castelnuovo-Mumford regularity, Proc. AMS. Vol. 130, N. 5, (2001), 1265-1274.
- Valla G.: Hilbert functions of graded algebras, in Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., 166, Birkhuser, Basel, 1998, 293-344.